A STUDY OF DIGITAL FILTER-OBSERVER SYSTEMS USING THE SECOND METHOD OF LYAPUNOV

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THESIS

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ABSTRACT

The basic filter-observer equations of Kalman for optimal and suboptimal filters are studied using the concepts of Lyapunov functions and stability theory. The Second Method of Lyapunov is used to form a basis for comparison of the convergence rates of such filters. Lyapunov functions are also used to derive constraining relations for the elements of the filter gain matrix leading to design criteria for sub-optimal filters. A derivation of the optimal filter gain based upon the Lyapunov function of a random variable is given. This derivation shows that the optimal filter converges most rapidly. A design of a sub-optimal filter for one class of signal models is given based solely upon stability constraints.

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SYMBOLS AND ABBREVIATIONS

X	an n square matrix unless otherwise specified
x	an n vector. Scalars will be specified unless obvious from the notation
t	independent variable time and is always a scalar (similarily for n in discrete time)
$\det(A)$	determinant of matrix A
$\mathrm{tr}(A)$	trace of matrix A
$\lambda(A)$	eigenvalue of matrix A
$\|\|\cdot\|\|$	norm of the argument (see Appendix A)
$\|\alpha\|$	magnitude of the complex scalar α
$N(x_e,\delta)$	neighborhood about the point x_e of radius δ. Taken as the usual sphere definition all x such that $\|\|x - x_e\|\| \leq \delta$
$E[x]$	expected value of the random vector x
ε	is a member of, belongs to
$\inf[\cdot]$	greatest lower bound

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I. INTRODUCTION

For many problems in control it is desirable to have some measure of all states in a given signal process. By their nature, many processes only allow us to physically measure or observe some of the states characterizing the process. Moreover, there is some amount of uncertainty in the measured states which may be characterized by additive measurement noise. Thus, estimation theory, as a device for estimating the states of a system in the presence of noisy and incomplete observations, has emerged as an important part of modern control theory.

In the early 1960's Kalman [K3,K4] advanced the well known Wiener filter theory [ref. D2,P1] by showing how such linear least square filters, which Wiener synthesized using classical frequency domain theory, may be realized in the time domain. This development coincided with the development of the digital computer and its use as an active component in process control systems.

The basic filter equation of Kalman is really a recursive weighted average technique of the following form:

$$\hat{X} = X + G\,(X-Z) \qquad (1)$$

where $\hat{X}$ is the corrected estimate, X is the projected estimate following known signal dynamics, G is the filter gain weighting factor, and Z is the signal observation including noise. G is generally time varying and its selection constitutes the basic design and nature of the filter. The difference equation of (1) becomes a differential equation in continuous time.

The corrected estimate, $\hat{X}$, is usually an n vector of the states of the signal process, and the observation, Z , is an m vector with $m \leq n$.

The selection of the filter gain has two objectives:

1. To provide an estimate of unobserved states based upon data from the observed states.
2. To provide for some weighting of the observations to allow compensation for the measurement noise.

Objective 1 is the basic observer problem which has emerged in recent years (such as the Luenberger observer [L1]) and does not consider any effects of noisy measurements. Objective 2, as stated, involves estimation theory and the stochastic properties of the observation noise as well as the random input signals which drive the system.

Current literature contains many examples of the application of Kalman filters to various problems. The computational capability required to implement such filters is generally large, and often a much simpler implementation works nearly as well. The literature also indicates that the method generally employed to determine the parameters of these filter implementations involves large scale simulation trials.

This thesis studies the properties of the basic filter observer equation in both continuous and discrete time using the Second Method of Lyapunov. Lyapunov theory [ref. K1,H1] was developed for investigating the stability properties of systems. Its application to the filter-observer problem has appeared in recent literature (Deyst and Price [D1]). However, this approach has not been fully exploited in the literature, and its possibilities are investigated in this thesis. This study leads to engineering insight into the operation of suboptimal

filters and provides some basis for comparison of the performance of these filters when applied to the same problem. Some specific results obtained include constraining relations on the filter gain matrix which insure:

1. that the filter is asymptotically stable.
2. that the filter will converge to steady state at a rate faster than a given bound.

Application of the filter observer equations as discussed here is limited to linear systems and it is assumed that the signal dynamics are known exactly. A review of stability concepts for linear time-varying systems and the Second Method of Lyapunov is included.

Chapter II reviews basic definitions and stability theorems for linear systems, based upon the concepts of state-space theory, with emphasis on forced systems so that these results may be applied readily to the filter-observer problem.

Chapter III reviews the Second Method of Lyapunov. This chapter includes some theorems on the determination of Lyapunov functions for certain classes of systems as well as some theorems on transient estimation using Lyapunov theory which have not appeared in the literature previously.

Chapter IV applies stability theory to the filter-observer homogeneous dynamics and contains many original results including the formulation of a time invarient Lyapunov function for the filter. The same Lyapunov function may be used for a large class of filters applied to the same problem. Particular filters have different time derivatives of the Lyapunov function. This leads to a method of comparing convergence rates for various filters applied to the same

problem. By applying the Hadamard-Gerschgorin theorem to provide sufficient conditions for positive definiteness of a symmetric matrix, certain constraining relations are derived for the elements of the filter gain matrix which lead to the design of stable filters.

Chapter V considers the forced filter dynamics. New results include the introduction of the concept of a Lyapunov function for a random variable. A derivation, using these Lyapunov functions, of the optimal filter is given showing that such filters also converge most rapidly to the minimum covariance of error.

Chapter VI applies the results of Chapter IV to the design of a class of suboptimal filters for the general n^{th} order signal model in phase variable form.

II. STABILITY OF LINEAR SYSTEMS

A. INTRODUCTION

A review of stability definitions and theorems are set forth in the following. Many current references are available for this material (H1, K1, K2, O1, S1, T1 and others). Although here we are mainly interested in linear systems, many of the theorems, as noted, apply to nonlinear systems. The reader is referred to Appendix A for a summary of norms and norm properties.

<u>Definition 2.1</u> Free system: Any system with no input forcing functions. Such a system is described as $\dot{x} = f(x,t)$

<u>Definition 2.2</u> Autonomous System: A free system which is also time invariant, (i.e., $\dot{x} = f(x)$)

<u>Definition 2.3</u> Equilibrium state(x_e): Any state for which $\dot{x} = 0$. For a free system $f(x_e,t) = 0$

<u>Definition 2.4</u> Boundedness: An equilibrium state x_e is said to be bounded if, and only if, there exists a neighborhood $N(x_e,\delta)$ such that, if the initial state $x(t_o)$ lies within N, then $||x(t)-x_e|| \leq m < \infty$ for all $t > t_o$

<u>Definition 2.5</u> Stability (in the sense of Lyapunov): An equilibrium state x_e is said to be stable, if, and only if, to any neighborhood $N(x_e,\varepsilon)$ there corresponds a neighborhood $N(x_e,\delta)$ such that, if $x(t_o)$ lies in $N(x_e,\delta)$ then $x(t)$ lies in $N(x_e,\varepsilon)$ for all $t > t_o$.

The difference between the last two definitions is important for nonlinear systems although the definitions are equivalent for linear systems. Note that Definition 2.5 requires that the state remain within a preassigned neighborhood (which may be arbitrarily small) for all $t > t_o$ whereas the definition of boundedness requires only that the distance (in state space) from x_e to the trajectory $x(t)$ remain bounded. Thus, boundedness allows for such things as limit cycles.

Definition 2.6 Asymptotic Stability: An equilibrium state is said to be asymptotically stable if

(1) It is stable.

(2) $||x(t)-x_e||$ approaches zero as t approaches infinity for any initial condition lying in the neighborhood $N(x_e,\delta)$ for which it is stable.

The above definitions are for strictly local conditions about the equilibrium state. If the neighborhood $N(x_e,\delta)$ can be the entire state space, then the system is said to be globally stable or asymptotically stable in the large (ASIL) about the equilibrium state x_e.[1]

Uniform stability is also a useful concept. It means that the neighborhood in which the initial state must lie is independent of the initial time. That is, δ is not a function of t_o.

Definitions 2.1 through 2.6 above also apply to discrete time systems with the use of (n,n_o) in place of (t,t_o) respectively. In the sequel, theorems and concepts are developed for both continuous and discrete time systems. The discrete notation is as follows:

---

[1] Some authors distinguish between global and stability in the large (H1) but for linear systems, they are identical.

The System:

$$x(n+1) = A_D(n)\, x(n) + B_D(n)\, u(n) \tag{2.1}$$

The Observation:

$$y(n) = H_D(n)\, x(n) \tag{2.2}$$

The Fundamental Matrix:

$$\Phi(n,n_o) = A_D(n-1)\, A_D(n-2) \ldots A_D(n_o) \tag{2.3}$$

The Solution:

$$x(n) = \Phi(n,n_o)\, x(n_o) + \sum_{k=n_o}^{n-1} \Phi(n,k+1)\, B(k)\, u(k) \tag{2.4}$$

For time invariant systems we have $\Phi(n) = A_D{}^n$.

Also it is interesting to note the relationship between continuous sampled systems and their discrete equivalent,for the time invariant case.

$$A_D = \Phi(T) \qquad B_D = \int_0^T \Phi(T-\tau)\, B\, d\tau \tag{2.5}$$

B. FREE SYSTEMS

It is well known that for linear time invariant systems, if the eigenvalues of the system matrix A all have negative real parts, then the system is asymptotically stable. Similarly for discrete systems the system is asymptotically stable to the origin if, and only if, the magnitude of the eigenvalues of A_D are less than unity. Such conditions are not easily established for time varying systems. However, the following theorems apply for the system $\dot{x}(t) = A(t)\, x(t)$, with $x_e = 0$ and the equivalent discrete case [ref. S1].

Theorem 2.1 The origin of a linear system is bounded if, and only if, the norm of the fundamental matrix $\Phi(t,t_o)$ (or $\Phi(n,n_o)$) is bounded.

To prove sufficiency note that

$$||x(t)|| = ||\Phi(t,t_0)\, x(t_0)|| \leq ||\Phi(t,t_0)||\; ||x(t_0)||$$

If $\quad ||\Phi(t,t_0)|| \leq K$

Then $\quad ||x(t)|| \leq K||x(t_0)||$

By definition 2.4, with $\delta = ||x(t_0)||$ it follows that the origin (x_e=0) is bounded. Necessity is shown by considering

$$||x(t)|| = ||\Phi(t,t_0)\, x(t_0)||$$

and noting that if the norm of $\Phi(t,t_0)$ is not bounded then some element of $\Phi(t,t_0)$ approaches infinity with time and hence $||x(t)||$ also approaches infinity and is unbounded.

<u>Theorem 2.2</u> Boundedness and stability of the origin are equivalent in linear systems.

To prove this, we must show that for any $\varepsilon > 0$ it is possible to find a δ such that if $x(t_0)$ is in $N(0,\delta)$ then $x(t)$ is in $N(0,\varepsilon)$ for all $t > t_0$. This reduces to showing that if $||x(t_0)|| < \delta$ then $||x(t)|| < \varepsilon$. By definition 2.4, if the origin is bounded we may take $\delta = \frac{\varepsilon}{K}$.

Then by theorem 2.1

$$||x(t)|| \leq K||x(t_0)|| < \frac{K\varepsilon}{K} = \varepsilon$$

which shows that boundedness implies stability. It is obvious that stability implies boundedness hence the two are equivalent.

<u>Theorem 2.3</u> The origin of a linear system is asymptotically stable if, and only if, the norm of $\Phi(t,t_0)$ is bounded for all $t_0 < t$ and $\Phi(t,t_0)$ approaches zero as t approaches infinity.

If $||\Phi(t,t_0)|| \leq K$ then the origin is stable (Theorems 2.1 and 2.2). We now have to show that $||x(t)||$ approaches zero as t approaches infinity. Now $||x(t)|| = ||\Phi(t,t_0)\ x(t_0)||$ and if $\Phi(t,t_0)$ approaches zero then $||x(t)||$ approaches zero.

Moreover, if $||x(t)||$ approaches zero for all $||x(t_0)||$ in $N(0,\delta)$ then $\Phi(t,t_0)$ approaches zero, since $x(t_0)$ is arbitrary in $N(0,\delta)$.

It should be noted that if a linear system is asymptotically stable, then it is also ASIL, since the conditions for stability depend only upon the fundamental matrix and are independent of the initial conditions of the states.

The foregoing theorems also apply to discrete systems with the change in notation from $\Phi(t,t_0)$ to $\Phi(n,n_0)$.

C. FORCED SYSTEMS

Now we are interested in studying forced systems of the form

$$\dot{x}(t) = A(t)\ x(t) + B(t)\ u(t) \tag{2.6}$$

We are particularly interested in the effect of u(t) on the system, assuming it is at rest at $t=t_0$. Then the solution of (2.6) is

$$x(t) = \int_{t_0}^{\tau} \Phi(\tau,t_0)\ B(\tau)\ u(\tau)\ d\tau \tag{2.7}$$

Stability of forced systems is considered with respect to a given set of inputs $U = \{u(t)\}$. Generally, this set is taken to be the set of bounded inputs.

<u>Definition 2.7</u> Stability of a forced system. A forced system is stable with respect to U if, and only if, the states are bounded ($||x(t)|| \leq K$) for all u(t) in U and all $t \geq t_0$.

<u>Theorem 2.4</u> A forced system is stable with respect to the set of bounded inputs if

$$\int_{t_0}^{t} ||\Phi(\tau,t_0)\ B(\tau)||\ d\tau \leq K_1$$

To prove sufficiency note that

$$||x(t)|| \leq \int_{t_0}^{t} ||\Phi(\tau,t_0)B(\tau)u(\tau)||\ d\tau \leq \int_{t_0}^{t} ||\Phi(\tau,t_0)B(\tau)||\ ||u(\tau)||d\tau$$

If $||u(\tau)|| \leq K_2$ then $||x(t)|| \leq K_1K_2$.

Hence the state is bounded.

A consolidating theorem appears in Timothy and Bona [T1] and Kalman [K1] with proof given in [K1].

<u>Theorem 2.5</u> For the system described by equation (2.6) if

(i) $||A(t)|| \leq K_1$ for all t

(ii) $0 < K_2 < ||B(t)v|| \leq K_3$ for all $||v||=1$

where v is an arbitrary vector, then the following statements are equivalent.

1. Any uniformly bounded input $||u(t)|| \leq K_4$ results in a bounded output $||x(t)|| < K_5 < \infty$
2. For all $t > t_0$

   $$\int_{t_0}^{t} ||\Phi(t,\tau)||d\tau \leq K_6 < \infty$$

3. The free system is uniformly asymptotically stable.

III. THE SECOND METHOD OF LYAPUNOV

A. INTRODUCTION

The foregoing indicates that stability for linear systems is completely determined by the boundedness properties of the fundamental matrix and the inputs to the system. This means in essence that the complete solution to the system must be obtained before stability can be ascertained. Lyapunov proposed a method by which stability of a system may be determined without finding the solution to the system equations. His method is based upon a generalized energy concept. From physics it is known that for a nonconservative system, if the total energy is always decreasing, then the system is stable. The total energy of such a system is a scalar function of the state of the system. Lyapunov has shown that the very existence of scalar functions of the state of a system which obey certain properties is sufficient to conclude that the system is stable. These functions are called Lyapunov functions. It is interesting to note that Lyapunov functions for a stable system are not unique and that a sufficient condition for stability is the existence of any such function.

Definition 3.1 Lyapunov function. Any function, $V(x)$, having the following properties is called a Lyapunov function.

(a) $V(x)$ is a continuous scalar function of the systems state vector and has continuous first partial derivatives.

(b) $V(x)$ is positive definite.

(c) $\dot{V}(x) = [\text{grad } V(x)]^T x$ is negative definite.

Definition 3.2 A scalar function, $V(x)$, of a vector argument is positive definite if, and only if,

(a) $V(0) = 0$

(b) $V(x) > 0$ whenever $x \neq 0$

Positive semi-definite means that equality is also allowed in part (b).

For negative definite functions the inequality is reversed.

In two dimensional state space the Lyapunov function is a cupped shaped surface setting on the origin. Its continuity and definiteness mean that the level curves of V projected on the state plane are closed about the origin and the curve $k_1 = V(x)$ is inclosed within $k_2 = V(x)$ if $k_1 < k_2$.

Under the foregoing conditions we can consider that the Lyapunov function gives a measure of distance[2] from the origin in state space as the system follows the trajectory $x(t)$.

B. THE BASIC STABILITY THEOREM

Before stating the theorem, consider the effect of explicit time variation upon the properties of the Lyapunov function. Suppose we have a function $V(x,t)$ and its time derivative

$$\dot{V}(x,t) = \frac{\partial V}{\partial t} + [\text{grad } V]^T \dot{x} \tag{3.1}$$

Now, $V(x,t)$ will be positive for all t if it is always greater than a positive function $\Theta(||x||)$. However, this is not sufficient to guarantee asymptotic stability even if $\dot{V} < 0$ for all t and $x \neq 0$.

---

[2]Distance used in this sense is not the same as the usual Euclidian distance. Distance is implied by the ordered nature of the level curves of V as discussed.

Since the grad V term in equation (3.1) indicates the motion of the system state (i.e., is multiplied by $\dot{x}$), it is possible that $\frac{\partial V}{\partial t}$ be sufficiently negative so that $\dot{V} < 0$ for all t, while the state moves outside of any bounding region $(N(x_e, \varepsilon))$. Hence, the state may never go to the equilibrium state x_e. Hsu and Meyer [H1] have a good discussion and Kalman and Bertram [K1] provide a rigorous proof of the following theorem.

Theorem 3.1 Consider the continuous time, free dynamics system $\dot{x} = f(x,t)$ where $f(0,t) = 0$ for all t. Suppose there exists a scalar function $V(x,t)$ with continuous first partial derivatives such that $V(0,t) = 0$ and

(i) there exists a continuous, non-decreasing scalar function α such that $\alpha(0) = 0$ and $V(x,t) \geq \alpha(||x||)$ for all t and $x \neq 0$.

(ii) there exists a continuous positive scalar function γ such that $\gamma(0) = 0$ and $\dot{V}(x,t) \leq -\gamma(||x||)$

(iii) $V(x,t) \leq \beta(||x||)$ where β is a continuous nondecreasing function such that $\beta(0) = 0$ for all t.

(iv) $\alpha(||x||) \to \infty$ with $||x|| \to \infty$

Then the system is asymptotically stable in the large to the equilibrium state $x_e = 0$ and $V(x,t)$ is a Lyapunov function for the system.

Requirement (i) insures that the Lyapunov function is always positive whereas (iii) insures that the function does not stay infinitely large throughout the state space. In particular it is required that the V function does, in fact, go to zero uniformly with time at the origin.

The γ function in (ii) assures that $\dot{V}$ is always negative. This together with (i) and (iii) assure that the Lyapunov function following the system trajectory is decreasing and must end up at the origin. To insure that the conditions exist for all of state space, requirement (iv) is imposed and the system is ASIL. It should be noted that a positive definite quadratic form for V automatically satisfies the requirements of (i), (iii), and (iv).

For discrete systems we define a Lyapunov function as V(x,n) and the rate of increase of V along the system trajectory as

$$\Delta V(x,n) = V(x(n+1),n+1) - V(x(n),n) \qquad (3.2).$$

With these changes, Theorem 3.1 becomes the corresponding theorem for discrete systems.

C. THE LINEAR AUTONOMOUS SYSTEM

For linear autonomous systems, it is a necessary and sufficient condition for ASIL that a quadratic form Lyapunov function exist. This is embodied in a theorem originally proved by Lyapunov.

<u>Theorem 3.2</u> The origin of a linear autonomous system, is asymptotically stable if, and only if, for any symmetric positive-definite matrix, C, there exists a unique positive-definite matrix Q which satisfies the matrix equation

$$A^TQ + QA = -C \qquad (3.3)$$

A full proof of this theorem appears in Hahn [H2]. Essential parts also are shown by Bellman [B1]. The major results of interest are delineated here for convenience.

1. The Lyapunov function is $V = x^TQx$

2. By direct computation,

$$\dot{V} = \dot{x}^T Qx + x^T Q\dot{x} = x^T A^T Qx + x^T QAX$$
$$= -x^T Cx \qquad (3.4)$$

3. Given any positive definite matrix C, then Q is uniquely determined by the matrix equation (3.3) if no two eigenvalues of A sum to zero or no eigenvalue is zero. (see [B1].)

4. If A is stable then the unique Q is given by

$$Q = \int_0^\infty e^{A^T t} \, C \, e^{At} \, dt \qquad (3.5)$$

Using 4 and a theorem[3] stated in Browne [B2] it is easy to show that if C is positive definite, then Q must be also. Since $C = B^T B$ where B is a real non-singular matrix[3] we can write

$$Q = \int_0^\infty e^{A^T t} B^T B e^{At} \, dt = \int_0^\infty (Be^{At})^T \, (Be^{At}) \, dt \qquad (3.6)$$

But B and e^{At} are non-singular. Hence Be^{At} is non-singular and Q is positive definite by reapplication of Browne's theorem[3].

Generally the use of Theorem 3.2 proceeds as follows:

(i) Choose any matrix C which is positive definite and solve the $n(n+1)/2$ algebraic equations (3.3) for the elements of Q.

(ii) Test Q for definiteness. If it is positive definite the system is asymptotically stable. If it is not positive definite the system is not asymptotically stable. This follows from the necessary and sufficient conditions stated in Theorem 3.2.

---

[3] From Browne; If C is any real non-singular matrix and $A = C^T C$ then A is positive definite. The converse is true, any positive definite matrix A can be expressed as such a product. Similarily, if C is n square of rank $r<n$ then A is positive semi-definite of rank r.

The theorem corresponding to Theorem 3.2 for discrete time systems is as follows [ref. K1,O1].

Theorem 3.2D The discrete time system described by

$$x(n+1) = A_D\, x(n)$$

is asymptotically stable if, and only if, for any given positive definite matrix C there exists a unique positive definite matrix Q satisfying

$$A_D^T Q A_D - Q = -C \qquad (3.7)$$

where $V = x^T Q x$ is a Lyapunov function with $\Delta V = -x^T C x$.

An algorithm for numerical solution of equation (3.7) for Q given any positive definite matrix C is derived in Appendix B. This is a new result and makes the application of Theorem 3.2D much easier than in the past.

At this point it is important to note that we cannot in general choose any positive definite matrix for Q and then perform the multiplication and addition indicated in (3.3) to obtain a C matrix which is positive definite, even if A is known to be stable. The following example illustrates this.

Example 1. For $A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$ and choosing $Q = I$,

it follows that

$$C = -(A^T + A) = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}$$

The system is stable, but C is indefinite. Therefore, $x^T x$ is not a Lyapunov function for the system described by the given transition matrix.

The following theorem has not generally appeared in the literature and is formulated here as a new result.

<u>Theorem 3.3</u> If, for a stable system, its transition matrix A is real and commutes with its transpose (i.e., $AA^T = A^TA$) then

$V = ||x||^2$ (i.e., Q = I)

is a Lyapunov function for the system described by A.

This theorem gives a sufficient condition for the square of the norm of the states to be a Lyapunov function. To show this we take $||x|| = (x^Tx)^{1/2}$ hence $V = x^Tx$ and show that $A^T + A$ must be negative definite. But $A^T + A$ is a real symmetric matrix and must have real negative eigenvalues [B2] if it is to be negative definite. But this is to say that the linear system described by the transition matrix $A^T + A$ is stable. Its fundamental matrix is then $\Phi = e^{(A^T+A)t}$. Now if $A^TA = AA^T$ then we can write $\Phi = e^{A^Tt} e^{At}$. By Theorems 2.1 and 2.2 of Section IIB, if $||\Phi|| \leq k$ then the system is stable. But since A is stable $||e^{At}|| \leq k_1$ so we write $||\Phi|| \leq k_1k_1 = k$.

For discrete systems the following theorem applies.

<u>Theorem 3.3D</u> If $A_D^T A_D = A_D A_D^T$ then a Lyapunov function for the discrete system $x(n+1) = A_D x(n)$ is $V = x^Tx$ with $\Delta V = x^T (A_D^T A_D - I) x$

Suppose there exists some transformation of states $x = Sy$ such that the transformed system matrix $D = S^{-1}AS$ is normal (i.e., $D^TD = DD^T$). Then we can apply Theorem 3.3 and a Lyapunov function for the transformed system is $V_y = y^Ty$. Since stability

properties for linear systems are invariant under a similarity transformation, we may apply the inverse transformation to V_y to find a suitable Lyapunov function in the x state space. The transformation must also be applied to $\dot{V}$. This is summarized in the following theorem.

Theorem 3.4 If a non-singular transformation exists for $x = S^{-1}y$ such that $D = SAS^{-1}$ is normal, then a Lyapunov function for the system $\dot{x} = Ax$ is:

$$V = x^T S^T S x \quad \text{with}$$

$$\dot{V} = x^T S^T (D^T + D) S x$$

$$= x^T (A^T S^T S + S^T S A) x$$

A similarity transformation may be applied to discrete systems as in the development of Theorem 3.4 for continuous systems.

D. LINEAR FREE SYSTEM

Here we look briefly at a system of the form $\dot{x} = A(t)x$. According to Theorem 3.1 of this chapter, if we can find a Lyapunov function for the system it may be time varying. However, we may also be able to find a Lyapunov function which is not explicitly a function of time. Assuming that we can find a positive definite Q such that $V = x^T Q x$ is a Lyapunov function then

$$\dot{v} = x^T [A^T(t)Q + QA(t)] x = -x^T C(t) x$$

In order to apply Theorem 3.1, we must find a positive function $\gamma(||x||)$ such that

$$\dot{V}(x,t) \leq -\gamma(||x||) \quad \text{for all } t.$$

Apparently we must somehow remove the time variation of A(t).

This obviously depends upon the nature of A(t). A general pursuit of this problem is not very fruitful. However, for the special case considered next we can get some useful results.

E. LINEAR SYSTEM WITH PERTURBED COEFFICIENTS

The special case of the linear free system considered here is that the state transition matrix A(t) is of the form $A(t) = A + G(t)$.

Now we assume $V(x) = x^TQx$ and compute

$$\dot{V}(x,t) = -x^TCx + x^T(G^T(t)Q + QG(t))x \tag{3.9}$$

where $-C = A^TQ + QA$.

From the results in section 3C, we know that if the system $\dot{x} = Ax$ is stable then for any choice of a positive definite matrix C we can compute a positive definite matrix Q. Moreover, if G(t) tends to zero as t increases, then there exists some t_0 such that for every $t > t_0$ the first term in the expression for $\dot{V}(x,t)$ will dominate and $\dot{V}$ is negative definite. Hence, the assumed quadratic form for V(x) is indeed a Lyapunov function. In this case we do not have uniform stability since the result depends upon t_0.

<u>Example 2</u>.

$$\text{Let} \quad A(t) = \begin{bmatrix} -1 + \frac{1}{t} & 0 \\ 0 & -2 + \frac{1}{t} \end{bmatrix}$$

$$\text{Then} \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad G(t) = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{t} \end{bmatrix}$$

Applying Theorem 3.3 page 25 we take $Q = I$ then $C = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

and C will dominate for all $t \geq 1/2$. Thus this system is ASIL for any $t_0 \geq 1/2$.

For the discrete time system

$$x(n+1) = Ax(n) + G(n)x(n)$$

we again assume a Lyapunov function $V = x^TQx$. Then by direct computation

$$\Delta V = V(x(n+1)) - V(x(n)) = -x^T(n)Cx(n) + x^T(n)QG(n)x(n)$$

where $-C = A^TQA - Q$.

From Theorem 3.2, page 22, we know that for any positive definite C we can find a positive definite Q if A is stable. Similar comments apply if G(n) tends to zero as for the continuous case.

F. FORCED LINEAR SYSTEMS

The usual application of Lyapunov's Second Method is to the study of the dynamics of homogeneous linear systems or to the study of the dynamics describing the motion of a system about some nominal trajectory usually attributed to the forcing function. In this section we consider a different approach which leads to a novel interpretation of the Lyapunov function with inputs present.

Consider the linear system

$$\dot{x} = Ax(t) + Bu(t) \tag{3.10}$$

Again assume a Lyapunov function $V = x^TQx$ and compute its time derivative.

$$\begin{aligned} \dot{V} &= \dot{x}^TQx + x^TQ\dot{x} \\ &= (u^TB^T + x^TA^T)Qx + x^TQ(Ax + Bu) \\ &= x^T(A^TQ + QA)x + u^TB^TQx + x^TQBu \end{aligned}$$

$$\dot{V} = -x^TCx + 2x^TQBu \tag{3.11}$$

From Section IIC we know that if the system $\dot{x} = Ax$ is stable we can find a positive definite Q for any given positive definite C such

that $-C = A^TQ + QA$. Also, note that the bilinear forms u^TB^TQx and x^TQBu are transposes of each other. Since they are scalars, they must be equal, and they may be combined as shown in the second term of equation (3.11).

By definition 2.1 in Section IIC page 13 the system described by (3.10) is stable if and only if $||x|| \leq k_1$. Thus if we can show that $\dot{V}$ given by (3.11) is negative for $||x|| > k_1$ we can be assured that the Lyapunov function will converge to a region determined by $||x|| \leq k_1$. This result is stated in the following theorem, which has not been stated explicitly in the literature heretofore.

<u>Theorem 3.5</u> For the system $\dot{x} = Ax + Bu(t)$, if the homogeneous system is ASIL and has a Lyapunov function $V = x^TQx$ with $\dot{V} = -x^TCx$ and $||u(t)|| \leq k_2$ for all t, then the system is stable and the state is sure to enter a region defined by

$$||x|| \leq k_1 = \frac{2k_2\,||Q||\,||B|| - \dfrac{\varepsilon}{k_1}}{\lambda_1} \qquad (3.12)$$

where $\lambda_1 = \lambda_{min}(C)$ the minimum eigenvalue of matrix C and $\varepsilon < 0$ with $|\varepsilon|$ arbitrarily small

Proof: The system is stable by definition 2.1 of Section IIC. Take as the system Lyapunov function $V = x^TQx$ then

$$\begin{aligned} \dot{V} &= -x^TCx + 2x^TQBu \\ &\leq -x^TCx + 2\,||x||\,||Q||\,||B||\,||u|| \end{aligned}$$

also since[4]

$$\lambda_1 \,||x||^2 \le x^T C x \tag{3.12a}$$

certainly $\dot{V} \le -\lambda_1 ||x||^2 + 2k_2 ||x|| \; ||Q|| \; ||B||$

To complete the proof we must show for all x such that $||x|| \ge k_1$

then $\dot{V} \le \varepsilon < 0$

When $||x|| \ge k_1$, then

$$||x|| \ge \frac{2k_2 ||Q|| \; ||B|| - \frac{\varepsilon}{k_1}}{\lambda_1}$$

and $\lambda_1 ||x||^2 \ge 2k_2 ||x|| \; ||Q|| \; ||B|| - \frac{\varepsilon ||x||}{k_1}$

Thus $0 > \varepsilon \ge \lambda_1 ||x||^2 + 2k_2 ||x|| \; ||Q|| \; ||B|| \ge \dot{V}$

It follows that $\dot{V}$ is negative when $||x|| \ge k_1$, indicating that the systems response is leaving the region.

The results of this theorem only give an upper bound for the region to which the state will converge in the steady state. Unfortunately, this bound depends upon the Lyapunov function chosen. However, the bound is determined without finding the solution to the system equations.

Example 3. Consider the stable system

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin \omega t$$

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

---

[4] Relation (3.12a) is derived using the Reyleigh Quotient $\frac{x^T C x}{||x||^2}$. See Bellman [B3] p. 110.

$$\lambda(A) = -1 \pm j \qquad AA^T = A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and compute a steady state bound for the state of the system.

Theorem 3.3, page 25 applies to the homogeneous dynamics, therefore a suitable Lyapunov function is

$$V = x^Tx \qquad \dot{V} = x^T(A + A^T)x = -x^TCx$$

that is, $\quad Q = I \qquad C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

and the homogeneous system is ASIL. The results of Theorem 3.5 are used to compute an upper bound for $||x||$ in the steady state. The parameters for equation (3.12) are as follows taking for the matrix

$$M = (m_{ij}) ; \quad ||M|| = [tr(MM^T)]^{1/2} \qquad \text{(see Appendix A)}$$

$$||u(t)|| = ||\sin \omega t|| \leq k_2 = 1$$

$$||Q|| = 1 \qquad ||B|| = 1$$

$$\lambda_{min}(C) = 2$$

$\varepsilon < 0$ and $|\varepsilon|$ is arbitrarily small

It follows that

$$||x|| \leq k_1 = \frac{2k_2||Q||\ ||B||}{\lambda_1} = 1$$

It should be noted here that if $C = \lambda_1 I$ the computed bound k_1 is independent of λ_1. This follows because $-C = A^TQ + QA$. If for $C = I$ the resulting Q is Q_1, then for $C = \lambda_1 I$ the resulting Q is $\lambda_1 Q_1$. Thus λ_1 cancels out in the expression for k_1.

The same theorem holds for discrete systems with appropriate changes in notation.

It is important to note that, for linear systems, the forcing functions have no affect upon the asymptotic stability properties of the system. They do however, influence the behavior of the system in the steady state.

G. TRANSIENT ESTIMATION

Since the level curves of V give, in some sense, a measure of distance from the origin, $\dot{V}$ may be used to estimate the convergence rate of asymptotically stable systems. The following is taken from Kalman and Bertram [K1] with some minor additions.

Consider the obvious inequality:

$$\dot{V}(x,t) = \frac{\dot{V}(x,t)}{V(x,t)} V(x,t) \leq -\eta_1 V(x,t) \tag{3.13}$$

where η_1 is the minimum of the ratio $-\frac{\dot{V}(x,t)}{V(x,t)}$ in some region (radius r) of state space excluding the origin. In cases where the basic stability theorem applies (Theorem 3.1 page 21) we may take[5]

$$\eta_1 = \inf_x \left\langle \frac{\gamma(||x||)}{\beta(||x||)} \; ; \quad 0 < ||x|| \leq r \right\rangle \tag{3.14}$$

Integrating equation (3.13)observe that

$$\int_{t_0}^{t} \frac{\dot{V}}{V} dt \leq - \int_{t_0}^{t} \eta_1 \, dt$$

---

5 $\inf_y \langle f(y), \; 0 < ||y|| \leq r \rangle$ implies the greatest lower bound of f(y) for all y such that $0 < ||y|| \leq r$.

$$V(x(t),t) \leq V(x_0,t_0)\ e^{-\eta_1(t-t_0)} \tag{3.15}$$

The number, $\frac{1}{\eta_1}$, is an upper bound for the time constant describing the convergence of the Lyapunov function following the system trajectory. If the system is linear the Lyapunov function is a quadratic form and $\frac{2}{\eta_1}$ is the corresponding bound for the convergence of the system trajectory.

Similarly we can write $\dot{V} \geq -\eta_2 V$ where η_2 is a maximum of the ratio $-\frac{\dot{V}}{V}$, giving a lower bound for the system time constant.

Kalman [K1] has carried out the minimization to obtain η_1 foı the linear time invariant case. These results are stated here. If the system $\dot{x} = Ax$ is ASIL and has a Lyapunov function $V = x^TQx$ with $\dot{V} = -x^TCx$ we take

$$\eta_1 = \min_x \left\langle x^TCx,\ V=1 \right\rangle \tag{3.16}$$

The minimization leads to

$$\eta_1 = \lambda_{min}(CQ^{-1}) \tag{3.17}$$

Similarly

$$\eta_2 = \lambda_{max}(CQ^{-1}) \tag{3.18}$$

Figure 1 illustrates the relationship of $V(x(t))$ and those functions formed by the η bounds.

These convergence estimates depend upon the Lyapunov function chosen. However, there are certain cases where the estimates are identical to those obtained directly from the system matrix A. The following new result is stated as a theorem the proof of which uses a result indicated in Bellman [B1] involving the eigenvalues of rational functions of several matrices, as follows:

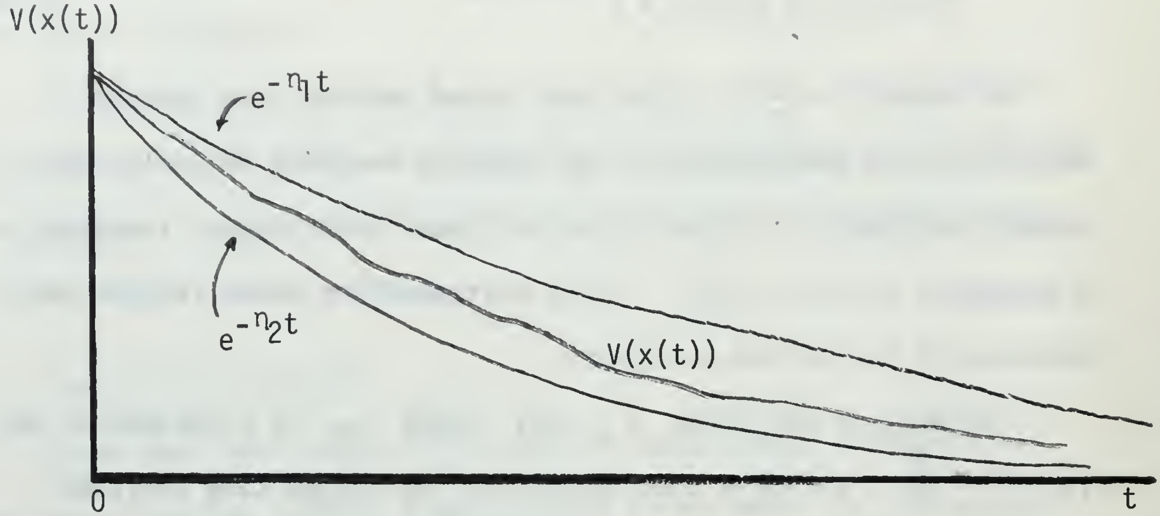


FIGURE 1

Lyapunov Function and convergence bounds following the system trajectory $x(t)$

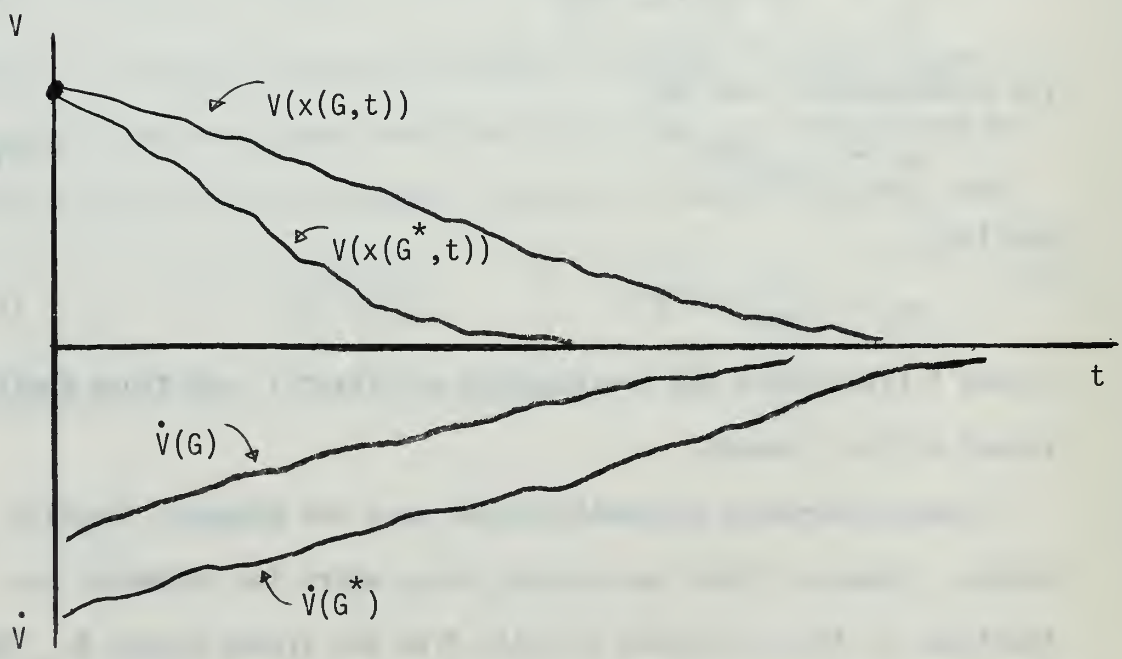


FIGURE 2

Effect of system parameters G on V. G^* minimizes $\dot{V}$

Let A and B be commutative matrices and $f(x,y)$ be a rational function. Then the characteristic roots of $f(A,B)$ are $f(a_i,b_i)$, where a_i and b_i are the characteristic roots of A and B in an ordered sequence which depends upon the matrices A and B.

<u>Theorem 3.6</u> If Theorem 3.3 of Section IIIC page 25 applies and Q is chosen as the identity matrix then $-\eta_1 = 2\lambda_{min}(A)$ and the convergence estimate using Lyapunov functions is identical with that of the actual system.

The proof is immediate since if $Q = I$ then $C = -(A + A^T)$. Taking $B = A^T$ in the above result and noting that $\lambda_i(A) = \lambda_i(A^T)$, we have that $\lambda_i(A + A^T) = 2\lambda_i(A)$. And the $\lambda_{min}(C) = 2\lambda_{min}(A)$.

The utility of this method is obviously not in estimating the convergence of linear time invarient systems since one eigenvalue problem is replaced by another. Its utility in this thesis is in application to time varying systems as illustrated in the following example.

<u>Example 4.</u>

Let $A(t) = \begin{bmatrix} -(1+\frac{1}{t}) & 0 \\ 0 & -(2+\frac{1}{t}) \end{bmatrix}$

It is desired to estimate the convergence rate for the system described by the given A(t). Write this as

$$A(t) = A - G(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{t} \end{bmatrix}$$

By applying the lemmas developed in Section IV C which follows, it can be shown that a suitable Lyapunov function is (see example 6 Section IV C)

$$V = x^Tx \qquad \dot{V} = -x^T(A + A^T)x - x^T(G(t) + G^T(t))x \qquad (3.19)$$

$$\leq -\gamma = -x^TCx \qquad \text{where } -C = A + A^T$$

For this example

$$Q = I \qquad C = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\eta_1 = \min_x \left\{ \frac{\gamma(||x||)}{V(x)} \right\} = \min_x \left\{ \frac{x^T Cx}{x^T x} \right\} = \lambda_{min}(CI^{-1}) = 2$$

Theorem 3.6 page 35 applies also, hence

$$\eta_1 = -2\lambda_{min}(A) = 2$$

Suppose that $\dot{V}$ is a function of a matrix of parameters G, as for example, the G(t) in equation (3.19). Now if for some $G = G^*$, $\dot{V}$ is a minimum with respect to G then the ratio $\frac{\dot{V}}{V}$ is also a minimum, hence, η_1, is maximum (see eq. (3.13)). It follows that if η_1 is maximum for $G = G^*$ then the system involving G^* converges faster than systems involving other G's. Figure (2) illustrates this concept of a minimum ratio $\frac{\dot{V}}{V}(G)$ which also appears in Section V in connection with the derivation of the optimal filter gain. In this case it is possible to choose G(t) so as to maximize the system convergence rate.

Corresponding results to the foregoing are established by Kalman in [K2] for discrete systems. The main difference is that a recursive relation for $V(x(n))$ is established as

$$V(x(n+1)) \leq e^{-2\eta T} V(x(n))$$

where T is the sample period and η is the corresponding constant in the continuous case. Again the minimization leads to

$$e^{-2\eta_1 T} = \lambda_{min}(CQ^{-1})$$

H. CONCLUSION

General results are difficult to obtain using the foregoing methods for time varying linear systems. Often the results give sufficient conditions only, and upper bounds which depend upon the chosen Lyapunov function. However, there is an advantage in that these results are always obtained easily without solving the system equation.

The approach of restricting the Lyapunov function to be explicitly time invariant was considered and serves as an introduction for a unified approach to the special system problem considered in the next chapter.

IV. THE HOMOGENEOUS FILTER OBSERVER EQUATION

A. INTRODUCTION

The remainder of this thesis is concerned with the study of the basic recursive averaging filter-observer using the foregoing concepts of stability theory.

The filter problem is usually formulated by considering a signal model driven by white noise and whose output measurements contain additive white noise. Such a model is described by

$$\dot{x} = Ax + w(t) \tag{4.1}$$

$$z = Hx + v(t)$$

where A is an nxn state transition matrix, H is an mxn measurement matrix, w(t) and v(t) are n and m vectors respectively of uncorrelated white noise.

Filtering is accomplished by recursively operating on the ovservations z to estimate the values of x. The estimates are denoted $\hat{x}$. The filter dynamics are given by

$$\dot{\hat{x}} = A\hat{x} + G(t)(z-H\hat{x}) \tag{4.2}$$

where G(t) is the nxm filter gain matrix the choice of which determines the filter characteristics. If the error, $e = x-\hat{x}$, is formed and the expected value of the error squared ($P = E[ee^T]$) is minimized to determine G(t) then (4.2) becomes the optimal integral squared error filter derived by Kalman and Bucy [K3]. If G(t) is determined by some other method, the filter is suboptimal (for example see S2). The optimal gain is given by

$$G(t) = P(t)H^TR^{-1} \tag{4.3}$$

where P(t) satisfies the matrix Ricatti equation

$$\dot{P} = AP(t) + P(t)A^T - P(t)H^TR^{-1}HP(t) + W \tag{4.4}$$

and $R = E[vv^T]$, $W = E[ww^T]$.

Suppose we rewrite equation (4.2) so that it is in the form of equation (4.1). We obtain

$$\dot{\hat{x}} = (A-G(t)H)\hat{x} + G(t)z \tag{4.5}$$

This equation is interpreted as a system having a state transition matrix, $A-G(t)H$, with the input vector z. Thus, we are indeed taking the observations z and filtering them to provide the output state vector $\hat{x}$.

Let us also find the equation for the error dynamics. Define the error, $e = x-\hat{x}$, to be the difference between the actual states and the estimated values out of the filter.

$$\dot{e} = \dot{x} - \dot{\hat{x}} = A\hat{x} + w(t) - Ax - G(t)(z-H\hat{x}) \tag{4.6}$$

Noting from (4.1) that $z = Hx + v(t)$ we write

$$\dot{e} = Ae - G(t)[Hx + v(t) - H\hat{x}] + w(t) \tag{4.7}$$

and $$\dot{e} = (A-G(t)H)e - G(t)v(t) + w(t) \tag{4.8}$$

Observe that the homogeneous dynamics for the filter equation (4.5) and the error equation (4.8) are identical. Both are described by the state transition matrix $A - G(t)H$. Consequently the results of this section apply to both the output of the filter and to the actual error.

Now suppose that G(t) is a matrix of constants, G_c. Then the filter transition matrix becomes $A - G_cH$ and the problem reduces to an ordinary linear time invariant case. Moreover, if $G = I$ the filter-observer equation reduces to that form derived by

Luenberger [L1][6] for observation of states.

In discrete time problems the signal model and optimal filter equations becomes respectively:

$$x(n+1) = \Phi(T)x(n) + w(n) \qquad (4.9)$$
$$z(n) = Hx(n) + v(n)$$

$$x(n) = \Phi(T)x(n-1) + G(n)[z(n) - H\Phi(T)x(n-1)] \qquad (4.10)$$

$$G(n) = P(n/n-1)H^T[HP(n/n-1)H^T + R]^{-1} \qquad (4.11)$$
$$P(n/n) = P(n/n-1) - G(n)HP(n/n-1)$$
$$P(n+1/n) = \Phi P(n/n)\Phi^T + W$$

where the notation $P(n+1/n)$ is the estimated covariance of error at time n+1 given the previous n corrected estimates.

For discrete systems, we find equations

$$\hat{x}(n+1) = (\Phi(T) - G(n)H\Phi(T))\hat{x}(n) + G(n)z(n) \qquad (4.12)$$

$$e(n+1) = (\Phi(T) - G(n)H\Phi(T))e(n) - G(n)v(n) + w(n) \qquad (4.13)$$

Again the unforced dynamics for the filter states and the estimation error are the same.

Bona [B4] has considered the problem of choosing the eigenvalues of the matrix $(\Phi(T) - G(n)H\Phi(T))$ to give a desired convergence property (namely, exact estimate, of the observation in a finite number of sample periods for noiseless systems) leading to the specification of $G(n)$. He has found that the observer can be made to converge as rapidly as desired. However, the observer is highly susceptible to error due to the presence of noise.

---

6 Luenberger [L1] has shown by using linear transformation theory that unobserved output states of a free linear system may be observed by passing the observed output states through another linear system whose transition matrix is A-H, provided A-H is stable.

The filter gain matrix G(t) or G(n) has a twofold purpose in the filter observer equation.

1. To provide observable outpus of unovserved states.
2. To provide some programmed weighting of the actual observations in an attempt to compensate for errors introduced by noise.

Further properties of G are studied in the following sections.

B. FILTER STABILITY

Stability of the optimal filter has been demonstrated in general by Kalman. A more general statement of the following theorem appears in Kalman and Bucy [K3].

<u>Theorem 1</u> (Kalman) [K3]. For a linear time invariant signal model also having stationary statistics, if the model is:

(i.) completely observable

(ii.) completely controlable

then the optimal filter is asymptotically stable.

It is evident that even if the signal model equation (4.1) section IV. A is unstable, the filter may still be asymptotically stable. That is, the matrix $(A-G(t)H)$ in equation (4.5) has its own stability properties. Thus, it is not necessary that the signal model be stable for the filter to be stable.

Deyst and Price [D1] have derived conditions for asymptotic stability of the discrete optimal filters by following the previous work of Kalman for the continuous optimal filter. Their method uses a time varying Lyapunov function namely choosing $Q = P^{-1}(n)$ where P(n) is the discrete covariance of errors. Then, by applying the conditions for controlability and observability, they found the desired bounding functions needed to apply the basic Lyapunov

stability theorem. The techniques developed in this paper, namely requiring the Lyapunov function to be time invariant, generally apply to a more limited class of signal models (i.e., asymptotically stable ones). However, the method is also much simpler and applies to suboptimal filters as well. Such a study leads to insight into what the filter gain must be to insure a stable filter and may in some cases be used to actually choose the G matrix requiring that the filter be stable.

Generally, the technique proposed here may be outlined as follows:

Assume that there is a Lyapunov function of the form $V = e^TQe$ then compute

$$\dot{V} = e^T(A^TQ + QA)e - e^T(H^TG^T(t)Q + QG(t)H)e \tag{4.14}$$

In some cases equation (4.14) may be used directly to determine the matrix Q. Whereas, in other cases, Q may be determined from $-C = A^TQ + QA$ by applying Theorem 3.2 page 22. If A is asymptotically stable then $\dot{V}$ of (4.14) becomes

$$\dot{V} = -e^TCe - e^T(H^TG^T(t)Q + QG(t)H)e \tag{4.15}$$

If equation (4.14) is negative definite for $t > t_0$ then the filter is ASIL by theorem 3.1 on page 22. Negative definiteness of $\dot{V}$ may be shown for all $t > t_0$. Alternately it is possible to find a bounding function, $\gamma(||x||)$, as required by theorem 3.1. Obviously, if the second quadratic form in equation (4.15) is at least positive semi-definite for all $t > t_0$ then, $\dot{V} \leq -e^TCe$ and the conditions of theorem 3.1 are met.

The fact that the filter is ASIL means that considering the filter as a linear system free of input forcing functions, the error starting at any arbitrary initial value will tend to zero as t increases.

This tells us nothing about the actual estimates or error. These questions must be considered with the input present which due to its stochastic nature, limits the asymptotic value of the effective error. Such consideration appears in Chapter V.

For discrete systems, taking the same Lyapunov function as for the continuous case and using Theorem (3.2D) page 24, we obtain:

$$\Delta V(e,n) = e^T[\Phi^T(I - G(n)H)^T Q(I - G(n)H)\Phi - Q]e \tag{4.16}$$

$$\Delta V(e,n) = -e^T ce - e^T(\Phi^T QG(n)H\Phi + \Phi^T H^T G^T(n)Q\Phi)e + e^T\Phi^T H^T G^T QGH\Phi e \tag{4.17}$$

and $$\Delta V = -e^T Ce - 2e^T(\Phi^T QG(n)H\Phi)e + e^T\Phi^T H^T G^T QGH\Phi e \tag{4.18}$$

when $$-C = \Phi^T Q\Phi - Q$$

The following serve to illustrate the kinds of filter problems considered in this thesis.

1. Filter Gain Constant

Case 1 — We are given $G(t) = G_c$ and G_c is a matrix of known constants. For this case we may take $\dot{V} = e^T Qe$ and rewrite equation (4.14) as

$$V = e^T((A-G_cH)^T Q + Q(A-G_cH))e \tag{4.18a}$$

Since A,H and G_c are all known, we may apply Theorem (3.2) page 22 directly to determine Q. An algorithm for numerical generation of Q is derived in appendix B for the discrete case (Theorem 3.2D).

Case 2 Choice of G_c so that the filter is stable. Now use equation (4.15) for $\dot{V}$ and determine Q based upon the signal model by applying Theorem 3.2 page 22. This requires that the signal model be asymptotically stable if a suitable Q is to be found.

The second quadratic form in equation (4.15) is now:

$$e^T(H^T G_c^T Q + QG_c H)e = e^T De$$

$$D = H^T G_c^T Q + QG_c H \tag{4.19}$$

If G_c can be chosen so that the matrix D in (4.19) is at least positive semi-definite(p.s.d.), the filter is sure to be stable since the first quadratic form in equation (4.15) is already negative definite.

Example 5

Consider again the system described in Example 3, page 30. Where

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Q = I \qquad C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Let $H = [1 \quad 0]$

In case 2, it is desired to choose G_c so that the matrix, D, in (4.19) is at least positive semi-definite, which is a sufficient condition to insure that the filter be stable. In order that H and G_c be conformable, G_c must be of the form

$$G_c = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

then

$$G_c H = \begin{bmatrix} g_1 & 0 \\ g_2 & 0 \end{bmatrix}$$

and from (4.19)

$$D = \begin{bmatrix} 2g_1 & g_2 \\ g_2 & 0 \end{bmatrix}$$

The condition that D be p.s.d. is that its principal minors be non-negative [ref. B2]. That is

$$2g_1 \geq 0$$

$$-g_2^2 \geq 0$$

Obviously $g_2 = 0$ and $g_1 \geq 0$ satisfy the conditions. Therefore, any filter with

$$G_c = \begin{bmatrix} g_1 \\ 0 \end{bmatrix} \quad \text{and} \quad g_1 \geq 0$$

will be ASIL.

Now suppose we are given

$$G_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Find the convergence bound η_1. From (4.18a)

$$\dot{V} = e^T(M^T Q + QM)e$$

where

$$M = A - G_c H = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix}$$

We may compute the Lyapunov function Q from theorem 3.2 for any p.d. matrix C such that

$$-C = M^T Q + QM$$

But in this case $MM^T = M^T M$ and we may use

theorem 3.3, page 25. Thus

$$Q = I \qquad C = -(M + M^T) = \begin{bmatrix} +4 & 0 \\ 0 & +2 \end{bmatrix}$$

The convergence bound, η_1, is given by (3.17)

$$\eta_1 = \lambda_{min}(CQ^{-1}) = \lambda_{min}(C) = 2$$

2. Special Time Varying Gain

Let the filter gain matrix be time varying such that

$$\lim_{t \to \infty} G(t) = 0 \tag{4.20}$$

the null matrix. Here we have a problem very similar to that discussed in section III E, and those comments apply here. Also if the matrix

$$(H^T G^T(t)Q + QG(t)H) \tag{4.21}$$

in equation (4.15) is at least positive semi-definite for all $t > t_0$, then we may not need the requirement of (4.20). Sufficient conditions for the matrix (4.21) to be positive semi-definite are developed in section IV C which follows.

3. The Optimal Filter

Substituting the optimal gain $G(t) = P(t)H^T R^{-1}$ into equation (4.15) we obtain

$$\dot{V} = e^T(A^T Q + QA)e - e^T(H^T R^{-1} HP(t)Q + QP(t)H^T R^{-1} H)e \tag{4.22}$$

If the signal model is asymptotically stable, Q may be determined for any p.d. matrix C as before and (4.22) becomes

$$\dot{V} = -e^T(C + H^T R^{-1} HP(t)Q + QP(t)H^T R^{-1} H)e \tag{4.23}$$

Now the condition for stability from (4.23) is that

$$f(y,t) = y^T(C + H^T R^{-1} HP(t)Q + QP(t)H^T K^{-1} H)y > 0 \tag{4.24}$$

for all t. Moreover, if

$$\min_t f(y,t) > 0$$

then the condition will hold for all t. From ordinary calculus a minimum occurs for $t = t_1$ if

$$\frac{df(y,t_1)}{dt} = 0 \qquad \text{and} \qquad \frac{d^2f(y,t_1)}{dt^2} > 0$$

Taking the derivative of equation (4.24) yields

$$\dot{f}(y,t) = y^T(H^TR^{-1}H\dot{P}(t)Q + Q\dot{P}(t)H^TR^{-1}H)y = 0 \qquad (4.25)$$

One way for (4.25) to be zero is for

$$\dot{P}(t) = 0 \qquad (4.26)$$

Kalman [K3] has shown that condition (4.26) yields an asymptotic or steady state value for P(t) in the limit as t approaches infinity by solving the Ricatti equation (4.4) page 39 for P with $\dot{P} = 0$. However, for this condition, $\lim_{t\to\infty} \ddot{P}(t)$ is also zero, hence, the foregoing does not in general lead to the desired minimum. Only in the trivial case where P(t) is diagonal does this method lead to the desired stability condition, since the diagonal elements of P(t) are monotonic decreasing to the limit.

The elements of the matrix in (4.25) are,in general, linear combinations of all the elements of $\dot{P}(t)$. There may be some $t = t_1$ which will make (4.25) zero other than in the limit as t approaches infinity. However, t_1 is generally difficult to find, even for a specific example. Often it is necessary to compute the trajectory for all elements of P(t); then, using this, study the properties of equation (4.24) at various values of time in an

effort to determine t_1. Besides being difficult, the method ceases to be useful, since, once P(t) is known for all t, complete knowledge of the filter is at hand.

Further research is required in the area of finding a suitable bounding function, $\gamma(||e||)$, for equations (4.22 or 4.23).

For suboptimal filters one might choose G(t) in equation (4.15) in order to establish the stability condition. The next section attempts to build some insight into the nature of G(t) required to insure a stable filter.

C. STABILITY CONSTRAINTS ON G(t)

In the last section we have found for the filter error dynamics that if $V = e^T Q e$ is a Lyapunov function then

$$\dot{V} = -e^T Ce - e^T(H^T G^T(t)Q + QG(t)H)e \tag{4.27}$$

$$\dot{V} = -e^T Ce - 2e^T(QG(t)h)e \tag{4.27a}$$

where $-C = A^T Q + QA$ and Q may be determined if the signal model $\dot{x} = Ax$ is ASIL. Equation (4.27a) is written as shown for convenience even though the second term is not a true quadratic form (i.e., the weighting matrix is not necessarily symmetric). In this section some sufficient conditions are found which insure that $\dot{V}$ in (4.27) be negative for all t. If these sufficient conditions hold for a given G(t) then the filter is shown to be ASIL. On the other hand, one may choose G(t) in order to make the conditions apply, thus designing a filter which is known to be ASIL.

In applying the basic stability theorem of section III B, we must find a positive function γ such that

$$\dot{V}(e,t) \leq -\gamma(||x||) \quad \text{for all } t$$

Equivalently, we may use a quadratic form for the bounding function of $\dot{V}$, that is, we must find a positive definite matrix, D, such that:

$$\dot{V}(e,t) \leq - e^T De = - \gamma \tag{4.28}$$

Applying this to equation (4.27) we require:

$$e^T[C + H^T G^T(t)Q + QG(t)H]e \geq e^T De \tag{4.29}$$

At this point we introduce inequality notation for symmetric matrices. Namely, $A \geq B$ means that (A-B) is positive semi-definite. If equality is removed, then (A-B) is positive definite. Writing relation (4.29) in this notation:

$$C + H^T G^T(t)Q + QG(t)H \geq D \quad \text{for all } t \tag{4.30}$$

The matrix D is important if one is to use the technique described in section 3G to estimate transient response of the filter. The form $\gamma = e^T De$ replaces $\dot{V}$ in the ratio $\dot{V}/V$ used to determine η_1 hence estimating the convergence rate of the filter. Large D means that the gradient of γ is steep, hence, the convergence bound is better since the actual filter converges faster than the estimate given by η_1. Thus, if G(t) is known, then it is desired to choose the largest D which will satisfy (4.30). On the other hand, if we wish to design the filter to converge faster than some given bound, then D is given and (4.30) provides a constraining relation for G(t).

If we rewrite (4.30) and apply the properties of norm to both sides, we get another bounding relation for G(t).

$$H^T G^T(t)Q + QG(t)H \geq D-C \tag{4.31}$$

$$||H^TG^T(t)Q|| + ||QG(t)H|| \geq D-C$$

$$2||Q||\ ||G(t)||\ ||H|| \geq ||D-C||$$

$$||G(t)|| \geq \frac{||D-C||}{2||Q||\ ||H||} \quad \text{for all } t. \tag{4.32}$$

Intuitively equality in (4.31) and (4.32) should result in a minimum condition of the left hand sides for all t. That is in (4.32)

$$\min_t ||G(t)|| = \frac{||D-C||}{2||Q||\ ||H||} \tag{4.32a}$$

A similar interpretation of the equality statement for (4.31) is not so easily conceived.

The basic requirement for stability given by (4.30) or (4.31) is that

$$\min_t \lambda_{min} (H^TG^T(t)Q + QG(t)H - D + C) = 0 \tag{4.33}$$

Application of (4.33) is very difficult. However sufficient conditions to insure that (4.33) does hold, may be found by applying an important theorem involving bounds on the eigenvalues of a matrix. The Hadamard-Gerschgorin theorem is stated here as found in Bodewig [B3].

Theorem (Hadamard-Gerschgorin).

> The eigenvalues of the complex matrix $A = [a_{ij}]$ lie inside the closed domain G consisting of all circles $K_i (i = 1,\ldots,n)$ with centers a_{i_i} and radii r_i where
>
> $$r_i = \sum_{j \neq i} |a_{ij}|$$

If we apply the H-G theorem to a real symmetric matrix and require that $a_{ii} \geq \sum_{j \neq i} |a_{ij}|$ for all i then we can get a sufficient condition test for positive semi-definiteness.

Lemma 1. Let A be a real symmetric matrix and if

$$a_{ii} \geq \sum_{j \neq i} |a_{ij}| \quad \text{for all } i, \text{ then A is positive}$$

semi-definite.

The proof is straight forward application of the H-G theorem. Since A is real symmetric its eigenvalues ($\lambda(A)$) must be real. Then applying the H-G theorem, the eigenvalues of A must lie between $\min_i (a_{ii} - r_i)$ and $\max_i (a_{ii} + r_i)$ where $r_i = \sum_{j \neq i} |a_{ij}|$ as illustrated in Figure 3a. But the hypothesis requires

$$a_{ii} \geq r_i = \sum_{j \neq i} |a_{ij}|$$

implying

$$a_{ii} - r_i \geq 0 \quad \text{for all } i$$

In particular, $\min_i (a_{ii} - r_i) \geq 0$,

which implies that the eigenvalues of A must be non-negative. Hence, A must be positive semi-definite and the lemma is proved.

Two lemmas will now be proven which will provide a method by which we may determine the best bounding matrix D.

Lemma 2. Let:

a. $M(t)$ be a real symmetric matrix of elements $m_{ij}(t)$

b. $m_{ii}(t) - \sum_{j \neq i} m_{ij}(t) = \alpha_i(t)$

c. $\alpha_i(t) \geq 0$ for all i and t

Then $M(t)$ is positive semi-definite for all t.

The proof is immediate from Lemma 1. Figure 3b illustrates the $\alpha_i(t)$.

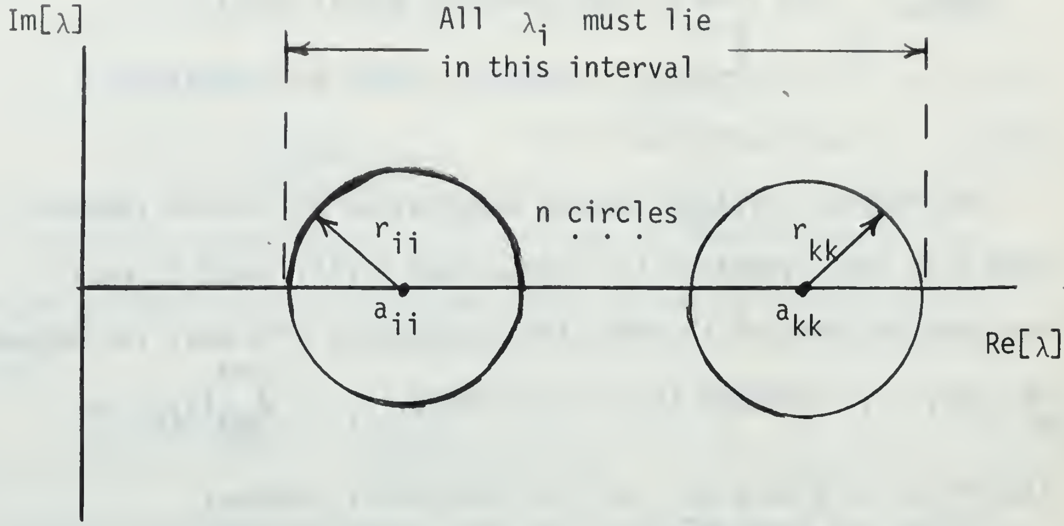


FIGURE 3a

Hadamard-Gerschgorin Theorem applied to a real symmetric matrix

$A = (a_{ij})$ $\qquad r_{ii} - \sum_{j \neq i} |a_{ij}|$

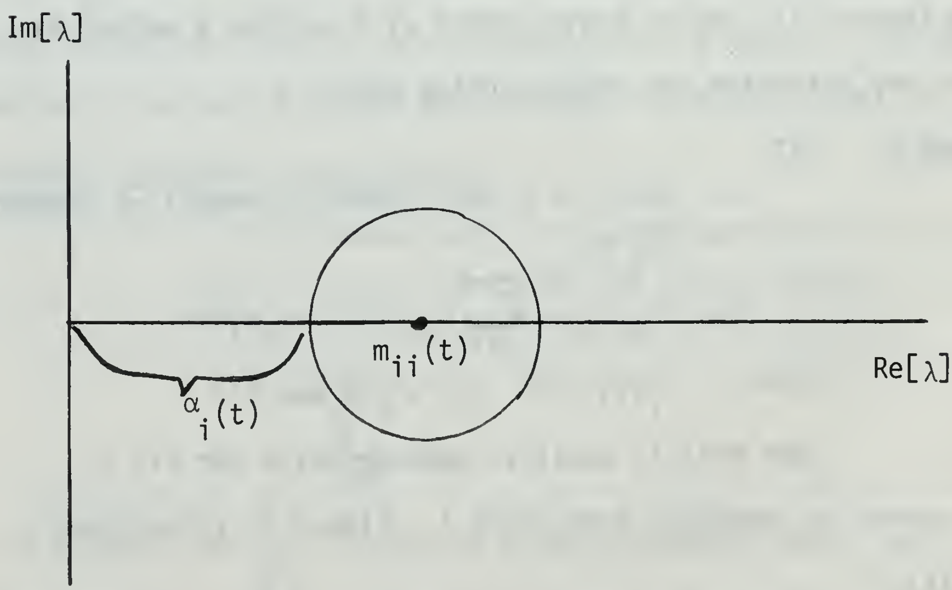


FIGURE 3b

Illustration of the $\alpha_i(t)$ of Lemma 2

As a result of lemma 2, if from equation (4.31) we set

$$M(t) = H^T G^T(t) Q + QG(t)H \tag{4.34}$$

then $$M(t) \geq D-C \quad \text{for all } t \tag{4.35}$$

If $\inf_t \alpha_i(t) = 0$ for any i then then condition for stability is obtained by choosing D = C. However, if $\alpha_i(t) > 0$ for all t, then we should be able to choose some $D > C$ giving rise to a tighter restriction on G(t).

<u>Lemma 3</u>. In Lemma 2 let:

a. $\alpha_k(t) = \min_i \alpha_i(t)$ for all t

b. $\alpha_k(t_1) = \min_t \alpha_k(t) > 0$

c. $m_{ii}(t) - m_{ii}(t_1) \geq \sum_{j \neq i} |m_{ij}(t) - m_{ij}(t_1)|$

Then $M(t) \geq M(t_1)$ with equality only at $t = t_1$.

The proof of Lemma 3 is by construction. Choose t_1 as specified in conditions (a) and (b), thus insuring that the smallest eigenvalue of M(t) will be positive for all t. For the matrix $(M(t) - M(t_1))$, use (b) in Lemma 2 to compute condition (c) in Lemma 3, and the lemma is proved.

It is apparent that the conditions in Lemma 3 may be used to insure that equation (4.33) holds. That is, by choosing M(t) as in (4.34) and the matrix $M(t_1)$ as in Lemma 3, we may rewrite equation (4.33) as an inequality since Lemma 3 only specifies a boundary for the eigenvalues of M(t). The result is

if $$\min_t \lambda_{min}(M(t) - D + C)) \geq 0 \tag{4.36}$$

then $$\lambda_{min}(M(t_1) - D + C)) \geq 0 \tag{4.36a}$$

Now if Lemma 3 applies, then we are assured that (4.36a) is true for some choice of the matrix D. However, we may also apply Lemma 3 directly to (4.31). This results in

$$M(t) \geq D - C \tag{4.37}$$

Since C is given and D may be computed directly, by replacing the left side of (4.37) with its minimum, $M(t_1)$, to establish the equality

$$M(t_1) = D - C \tag{4.38}$$

This follows, since by Lemma 3, $M(t) \geq M(t_1)$.

On the other hand, if D is given, as established by some convergence bound, η_1, then the conditions in Lemma 3 may be applied to (4.37), thus, establishing constraining relations on the elements of M(t).

We are really interested in the matrix G(t). From (4.34) it is obvious that every element of M(t) will in general be a linear combination of all the elements of G(t). This follows, since H and Q are determined by the signal model and are constant matrices. Thus, in general

$$M(t) = [m_{kl}] = [(\sum_{ij} b_{ij}\, g_{ij}\, (t))_{kl}] \tag{4.39}$$

This may not be difficult to apply to specific examples since H and Q are often very simple matrices and many of the $b_{ij} = 0$.

The following illustration refers to Example 4, page 35.

Example 6. As in Example 4, consider the system described by the transition matrix

$$A(t) = \begin{bmatrix} -(1 + \frac{1}{t}) & 0 \\ 0 & -(2 + \frac{1}{t}) \end{bmatrix}$$

This is equivalent to a filtering problem where the signal model is described as follows:

The system: $\dot{X} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

and observation $z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

where $A \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad H = I$

and w_1, w_2, v_1, v_2 are white Gaussian noise inputs. The homogeneous filter dynamics are given in (4.8) as

$\dot{e} = (A - HG(t))\, e = A(t)\, e$

where

$$G(t) = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{t} \end{bmatrix}$$

By using Lemma 2 we will derive the Lyapunov function given in equation (3.19) of Example 4.

First, assume a Lyapunov function of the form

$$V = e^TQe$$

then by equation (4.15)

$$\dot{V} = -e^TCe - e^T(H^TG^T(t)Q + QG(t)H)\, e$$

$$\leq -e^TDe$$

where Q is determined from $-C = A^TQ + QA$. But in this case $AA^T = A^TA$ and we may use Theorem 3.3, page 25 to find Q and C . The result is

$$Q = I \qquad C = -(A^T + A) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

Now in order to apply Lemma 2 we find from equation (4.34)

$$M(t) = G^T(t) + G(t) = \begin{bmatrix} \frac{2}{t} & 0 \\ 0 & \frac{2}{t} \end{bmatrix} .$$

This follows since $Q = H = I$.

Then by Lemma 2 we must have

$$\alpha_1 = \alpha_2 = \frac{2}{t} \geq 0 \qquad \text{for all } t.$$

Obviously, $\alpha \geq 0$ for any $t \geq t_o > 0$. Since $\inf_t \alpha(t) = 0$ in the limit as t approaches infinity, the condition for stability is obtained by choosing $D = C$. That is, $-e^TCe$ is a suitable bounding function for $\dot{V}$. Hence, $\dot{V}$ is negative definite and the filter is ASIL by Theorem 3.1, page 21.

Lemma 3 applies to problems where the $\alpha_i(t)$ in Lemma 2 are greater than zero for all t. Consider the slight extension of the above example by letting

$$G(t) = \begin{bmatrix} \frac{1}{t} + 1 & 0 \\ 0 & \frac{1}{t} + 2 \end{bmatrix}$$

By equation (4.34)

$$M(t) = \begin{bmatrix} 2 + \frac{2}{t} & 0 \\ 0 & 4 + \frac{2}{t} \end{bmatrix}$$

And in Lemma 3

$$\alpha_1 = 2 + \frac{2}{t} \qquad \alpha_2 = 4 + \frac{2}{t}$$

Take t_1 in the limit as t approaches infinity. Then from conditions (a) and (b) in Lemma 3:

$$\alpha(t_1) = 2 > 0$$

It follows that

$$M(t_1) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

and condition (c) Lemma 3 reduces trivially to Lemma 2. Using equation (4.38) we may compute the matrix D

$$M(t_1) = D - C$$

$$D = M(t_1) + C = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}$$

By using the results of Section III G we may compute a convergence bound for the filter employing the given $G(t)$.

The problem of using the foregoing Lemmas to obtain constraints on the elements of $G(t)$ giving rise to a class of suboptimal filters is deferred to Chapter VI.

The foregoing Lemmas apply to discrete systems simply by replacing t with n. Now $M(n)$ is specified from equation (2D) in Section 4B. Namely:

$$M(n) = \Phi^T Q G(n) H \Phi + \Phi^T H^T G^T(n) Q \Phi - \Phi^T H^T G^T(n) Q G(n) H \Phi \qquad (4.40)$$

However, (4.40) may be simplified. This follows since Φ is non-singular and appears as a congruent transformation on the inclosed matrix and does not affect the sign definiteness of the inclosed matrix [ref. B2]. So in effect we may take

$$M(n) = QG(n)H + H^T G^T(n) Q - H^T G^T(n) Q G(n) H \qquad (4.41)$$

D. CONCLUSIONS

In this section we studied the homogeneous dynamics of the basic filter-observer equation (4.5) and its error dynamics equation (4.8). Since these homogeneous dynamics are identical, the results of this section apply to both.

The basic approach in this section is to assume a time invariant Lyapunov function and compute its time derivative, $\dot{V}$, subsequently showing that $\dot{V}$ is negative definite for a given filter gain matrix G(t). This was found difficult to do in general terms. Often it is possible to determine the quadratic form for V based solely upon the signal model. This requires that the model be ASIL. It is desirable to choose the Lyapunov function based only upon the signal model if one wishes to compare the convergence of two different filters designed for the same signal model by using the estimation techniques described in section III G. Under these conditions the Lyapunov function for the two filters is the same, but the associated $\dot{V}$ is modified according to the particular filter (see equation (4.15)).

The techniques developed in this chapter cannot be readily applied to the optimal filter. This is left as an area for additional research.

Several constraining relations between the bounding γ function required for stability and characteristics of the gain matrix were developed in section IV C. In general these relations are difficult to apply, but in some simple cases they lead to a method of determining G(t) to insure a stable filter which converges faster than some desired bound.

V. THE FORCED FILTER EQUATION

A. INTRODUCTION

In this chapter the forcing functions are included in the filter dynamics. From the study of linear forced systems in section III F, recall that the forcing functions do not affect the asymptotic stability of the system dynamics. They do, however, determine the steady state properties of the system. Such is also the case for the filter-observer system, and these properties are investigated in this chapter.

The error dynamics for the filter given in equation (4.8) are repeated here

$$\dot{e} = (A-G(t)H)\, e - G(t)\, v(t) + w(t) \tag{5.1}$$

As in chapter IV, assume a Lyapunov function of the form

$$V = e^T Q e \tag{5.2}$$

then by direct computation using (5.1)

$$\begin{aligned} \dot{V} &= \dot{e}^T Q e + e^T Q \dot{e} \\ &= e^T(A^T Q + QA)\, e - e^T(H^T G^T(t) Q + QG(t)H)\, e \\ &\quad + 2w^T(t) Q e - 2v^T(t) G^T(t) Q e \end{aligned}$$

$$\begin{aligned} \dot{V} &= -e^T C e - e^T(H^T G^T(t) Q + QG(t)H)\, e \\ &\quad + 2w^T(t) Q e - 2v^T(t) G^T(t) Q e \end{aligned} \tag{5.3}$$

where Q is determined, as before, for any given p.d. matrix C by the signal model which is required to be ASIL.

It should be noted that the only difference between equation (5.3) and equation (4.15) is the addition of two terms involving the forcing functions $w(t)$ and $v(t)$. However, this particular

application is different because the forcing functions are random vectors. To consider this aspect we use the expectation operator[7] and rewrite equation (5.3). Since Q is completely determined for any choice of positive definite C, we write this equation in terms of Q. Applying the matrix identity (5.4) to (5.3)

$$x^T M y = tr[Myx^T] = tr[yx^T M] \tag{5.4}$$

results in

$$\begin{aligned} \dot{V} = \; & tr[Q(A-G(t)H)ee^T + ee^T(A-G(t)H)^T Q \\ & + 2Qew^T(t) - 2ev^T(t)G^T(t)Q] \end{aligned} \tag{5.5}$$

In order to remove the random nature from this problem we take the expectation of V and $\dot{V}$. The forcing functions w(t) and v(t) are uncorrelated white Gaussian random processes. Consequently, the estimate error is also a Gaussian random process. We define the error covariance matrix $P = E[ee^T]$. Thus the Lyapunov function (5.2) becomes:

$$E[V] = trE[Qee^T] = tr(QP) \tag{5.6}$$

The expectation operation is done over an ensemble of Lyapunov functions each of which is the result of one experiment. Taking the expectation and applying (5.4) to (5.5) results in

$$\begin{aligned} E[\dot{V}] = \; & tr[Q(A-G(t)H)P + P(A-G(t)H)^T Q] \\ & + 2QE[ew^T(t)] - 2E[ev^T(t)]G^T(t)Q] \end{aligned} \tag{5.7}$$

---

[7] The expectation operator is defined as $E[x] = \int_{-\infty}^{\infty} xf(x)\,dx$ where f(x) is the density function of x [ref. Papoulis, P1]

To finish this problem we must first determine what $E[ew^T]$ and $E[ev^T]$ are. Equation (5.1) has a solution of the form

$$e(t) = \Phi(t_1 t_o)e(t_o) + \int_{t_o}^{t} \Phi(t,\tau)w(\tau)d\tau - \int_{t_o}^{t} \Phi(t,\tau)G(\tau)v(\tau)d\tau \qquad (5.8)$$

But $w(t)$, $v(t)$ and $e(t_o)$ are uncorrelated so that

$$E[w^T w] = E[v^T v] = E[w^T e(t_o)] = E[v^T e(t_o)] = 0 \qquad (5.9)$$

Also define

$$E[w(\tau)w^T(t)] = W\delta\ (t-\tau)$$

$$E[v(\tau)v^T(t)] = R\delta\ (t-\tau) \qquad (5.10)$$

where $\delta(t-\tau) = \begin{cases} 1, & t = \tau \\ 0, & t \neq \tau \end{cases}$

Now using (5.8), form ew^T.

$$e(t)w^T(t) = \Phi(t,t_o)e(t_o)w^T(t) + \int_{t_o}^{t} \Phi(t,\tau)w(\tau)w^T(t)d\tau$$

$$- \int_{t_o}^{t} \Phi(t,\tau)G(\tau)v(\tau)w^T(t)d\tau \qquad (5.11)$$

Taking the expectation of (5.11), interchanging the order of integration and expectation and using (5.9),(5.10)

$$E[e(t)w^T(t)] = 0 + \int_{t_o}^{t} \Phi(t,\tau)W\delta(t-\tau)d\tau - 0$$

$$= 1/2\ [W] \qquad (5.12)$$

This follows from the sifting property of the Delta function[8].

Similarly we can find:

$$E[e(t)v^T(t)] = -1/2\ [G(t)R] \qquad (5.13)$$

---

8 The Delta function sifting property applied to an integration limit at the location of the Delta function is used here.

(i.e., $\int_a^b f(x)\delta(x-b)dx = 1/2\ [f(b)]$

Substituting (5.12) and (5.13) into (5.7) yields

$$E[V] = tr[Q(A-G(t)H)P + P(A-G(t)H)^TQ + QW + G(t)RG^T(t)Q] \tag{5.14}$$

Note that if $Q = I$ in (5.14) the result becomes the error dynamics developed by others. Specifically Athens and Tsi [A1] arrived at (5.14) in their derivation of the optimal filter and Sims and Melsa [S2] also arrived at this result for specific optimal filters.

For discrete systems a difference equation for $E[V(n)]$ analogous to equation (5.14) is easily obtained by using equation (4.13) to form

$$E[e^TQe] = tr[QP(n)] \tag{5.15}$$

similarly as for the continuous case. The result for $P(n)$ is

$$P(n) = (I-G(n)H)P'(n)(I-G(n)H)^T + G(n)RG^T(n) \tag{5.16}$$

$$P'(n) = \Phi P(n-1)\Phi^T + W$$

where R is the covariance matrix for the measurement noise and W is for the noise driving the signal model. A derivation of equation (5.16) also appears in Sorenson [S3].

B. DERIVATION OF THE OPTIMUM G(t)

In effect the stochastic Lyapunov function (5.6) for the filter-observer is a linear combination of all the covariance of errors between the estimates (outputs of filter) and the actual signal being estimated. The covariance of error gives a measure of how well the filter should be expected to perform. As such, the Lyapunov function (5.6) gives a scalar measure of the performance of the filter. It is desirable that the covariance of error be minimum for all t. Thus the Lyapunov function (5.6) may be used as a cost

criterion to be minimized. The optimum filter is then formed by choosing $G(t)$ such that $E[v]$ is minimized for all t. If (5.14) is negative for all $t \geq t_o$, then

$$E[\dot{V}] < 0 \tag{5.17}$$

and the filter is ASIL since $E[v] > 0$ for all t. Moreover, if $E[\dot{V}]$ is negative, then $E[V]$ is assured to be a minimum if the magnitude of its time derivative is maximum for all t. The optimum $G(t)$ can be derived as follows, using the concept of a gradient matrix. The following formulas for the partial derivative of a scalar function (tr) of several matrices with respect to one of the matrices can be proven (Athans and Tsi [A1]).

$$\frac{\partial\ tr(AB^T)}{\partial\ B} = \frac{\partial\ tr(BA^T)}{\partial B} = A \tag{5.18}$$

$$\frac{\partial\ tr(ACA^T)}{\partial A} = 2AC \tag{5.19}$$

where C is symmetric.

Taking the partial derivitive of (5.14) with respect to $G(t)$ using (5.18) and (5.19) yields

$$\frac{\partial\ E[\dot{V}]}{\partial G} = 0 = -\ 2PH^TQ + 2G(t)RQ$$

and solving for

$$G(t) = PH^TR^{-1} \tag{5.20}$$

This is the identical result derived by Kalman and others [A1, K3, S3] for the optimal filter. To complete the derivation of the optimal filter we must show that (5.20) results in a minimum for (5.14). Note the second partial derivative of (5.14) with respect to G, by applying (5.18) and the fact that R and Q are symmetric.

$$\frac{\partial^2 E[\dot{V}]}{\partial G^2} = 2QR \tag{5.21}$$

If $$QR = RQ \tag{5.22}$$

R and Q commute, then the right hand side of (5.21) is positive definite since Q and R are positive definite. This follows from the result indicated in Bellman [B1] given on page 33 of chapter III. Now if (5.21) is positive definite then (5.20) does result in a minimum for (5.14). But if (5.17) holds, then the magnitude of $E[\dot{V}]$ is maximum (for G in (5.20)), hence E[V] in (5.6) is a minimum.

Moreover, the ratio

$$\frac{-E[\dot{V}]}{E[V]} \tag{5.23}$$

is also maximum, and the following important conclusion is reached.

(1) The filter using the G(t) given by (5.20), is assured of having the most rapid convergence rate to the minimum covariance of error.

This follows from the discussion on page 36 of chapter III, section G.

The foregoing derivation strictly holds only if conditions (5.17) and (5.22) are true. However, the result indicated in (1) above does give engineering insight into the nature of optimal filters and to desired properties of suboptimal filters. That is, it is possible to make an engineering trade-off between convergence rate and steady state covariance of error in the design of suboptimal filters which are much easier to implement than the Kalman filter.

C. CONCLUSION

The concept of a Lyapunov function for random variables was introduced for the forced filter dynamics. A derivation of the

optimal filter gain matrix was given leading to the conclusion that the optimal filter is the most rapid converging to the minimum covariance of error. As a consequence, it is possible in the design of suboptimal filters to make an engineering trade-off between convergence rate and steady state estimation error.

VI. AN EXAMPLE OF A SUBOPTIMAL FILTER

A. INTRODUCTION

Consider the discrete form of a system described in phase variable form with only one state being observed. Namely:

$$\begin{aligned} \dot{x} &= Ax \\ y &= Hx \end{aligned} \qquad \text{where} \qquad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ . & . & . & & . \\ . & . & . & & . \\ . & & . & . & . \\ . & & & . & \\ & & & 0 & 1 \\ -a_n & & \cdots & & -a_1 \end{bmatrix} \qquad H = [1\ 0\ \ldots\ 0] \tag{6.1}$$

It is assumed that A is stable and H is a $1 \times n$ matrix, and that $\Phi(T)$ has been found as described in Section IIA where T is the sampling period. Note the transition matrix for the discrete filter (4.12)

$$\hat{x}(n+1) = (\Phi(T) - G(n)\, H\, \Phi(T))\hat{x}(n) + G(n)z(n)$$

If H is $1 \times n$, then $G(n)$ must be $n \times 1$ in order for the indicated multiplication to be conformable. This is indeed fortunate, since $G(n)$ is chosen to insure that the filter is stable by application of the lemmas in Section IV C, and these lemmas only give n constraining relations for the elements of G.

Proceeding, we determine the system Lyapunov function $V = e^TQe$ for $\Delta V = -e^TCe$ by finding Q such that $\Phi^TQ\Phi - Q = -C$ for a given positive definite C. This may be easily done using the algorithm developed in Appendix B.

Now, in order to insure stability, the matrix $M(n)$ in (4.41) must be positive semi-definite in accordance with Lemmas 1 and 2 developed in

Section IV C. The argument, (n), will be dropped for convenience in the following and the notation used for Q is:

$$Q = (b_{ij})_{n \times n} \qquad \text{and for} \quad G = (g_j)_{n \times 1}$$

Repeating equation (4.41) here

$$M = QGH + H^T G^T Q - H^T G^T QGH \tag{6.2}$$

Multiplying (6.2) out, noting that:

$$G = [g_1 \; g_2 \; . \; . \; . \; g_n]^T \qquad H^T G^T = \begin{bmatrix} G^T \\ \hline 0 \end{bmatrix} \tag{6.3}$$

$$GH = [G_i \mid 0]_{n \times n}$$

$$QGH = \left[\begin{array}{c|c} \sum_1^n b_{ij} \, g_j & \\ \vdots & 0 \\ \sum_1^n b_{nj} \, g_j & \end{array}\right]_{n \times n}$$

G^TQG is in fact a quadratic form in the vector G and

$$H^T H = \begin{bmatrix} 1 & 0 & . \; . \; . & 0 \\ 0 & 0 & & \\ . & & . & \\ . & & & . \\ 0 & . \; . \; . & & 0 \end{bmatrix}$$

then $$H^T G^T QGH = \left(\sum_{ij} b_{ij} \, g_i \, g_j\right) H^T H \tag{6.4}$$

Substituting the results, (6.3) through (6.4), into (6.2)

$$M = \begin{bmatrix} \left(\sum b_{1j} g_j - \sum_{ij} b_{ij} g_i g_j\right) & \sum b_{2j} g_j & . \; . \; . & \sum b_{nj} g_j \\ \sum b_{2j} g_j & & & \\ \vdots & & 0 & \\ \sum b_{nj} g_j & & & \end{bmatrix} \tag{6.5}$$

Now the α_i of Lemma 2 are:

$$\alpha_i = 0 - \left| \sum_{j=1}^{n} b_{ij} g_j \right| \qquad i = 2,3, \ldots ,n$$

According to Lemma 2, $\alpha_i \geq 0$ however, it is evident that for $i \geq 2$ we must choose $\alpha_i = 0$ since they can never be positive. This results in $n - 1$ linear equations in n unknowns depicted by:

$$\sum_{j=1}^{n} b_{ij} g_j = 0 \qquad i = 2,3, \ldots ,n \tag{6.6}$$

Choosing to solve these $n - 1$ equations for $g_2 \ldots g_n$ in terms of g_1 , rewrite equations (6.6) in matrix form

$$\begin{bmatrix} b_{22} & b_{23} & \cdots & b_{2n} \\ b_{32} & & & \\ \vdots & & & \\ b_{n2} & & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} g_2 \\ g_3 \\ \vdots \\ g_n \end{bmatrix} = \begin{bmatrix} -b_{21} \\ -b_{31} \\ \vdots \\ -b_{n1} \end{bmatrix} g_1 \tag{6.7}$$

The square matrix in (6.7) must have an inverse, since it is a principal minor of Q which is positive definite.[8] Thus, (6.7) may in fact be solved for $g_2 \ldots g_n$ in terms of g_1. The remaining relation necessary to determine g_1 is given by α_1 of Lemma 2.

$$\alpha_1 = \sum_{j=1}^{n} b_{1j} g_j - \sum_{i,j} b_{ij} g_i g_j - 0 \tag{6.8}$$

But α_1 must be non-negative, giving:

$$\sum_{j=1}^{n} b_{1j} g_j \geq \sum_{i,j} b_{ij} g_i g_j \tag{6.9}$$

---

[8]The determinants of the principal minors of a positive definite matrix are positive [ref. B2].

Using equation (6.7) and relation (6.9) a family of G matrices may be determined. Since some α_i is zero, the stability condition as discussed in Section IV C following Lemma 2 is obtained by choosing $D = C$. Referring to Section III G the convergence bound for this filter is:

$$e^{-2\eta_1} = \lambda_{min}(DQ^{-1}) = \lambda_{min}(CQ^{-1}) \tag{6.10}$$

But, (6.10) is the convergence bound for the signal model since C was chosen for the signal model. This means that the transient behavior of the filter is assured to be no worse than that of the signal model.

Unfortunately, the foregoing Lyapunov technique developed cannot be used directly for optimal filters involving the phase variable form of signal model. In particular, ΔV , equation (4.18), has not been shown to be negative definite for the optimal filter in this example. However, a reasonably good comparison of the suboptimal filter design and the optimal filter may be made by computing $tr(QP(n))$, equation (5.15) for both filters using the Q determined from the signal model. This follows because the Lyapunov function (5.15) is a scalar function of all the covariance of error for the filter.

B. NUMERICAL EXAMPLE

In order to summarize the foregoing results, a simple third order numerical example is given.

A suboptimal filter-observer is to be designed for the given signal model. The filter is required to be ASIL.

The signal model is described as follows. Given the signal dynamics described by the continuous time matrix in phase variable form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -17 & -12 \end{bmatrix} \tag{6.11}$$

the discrete version of this signal model for a sample period of $T = .15$ is driven by a white Gaussian random sequence whose covariance matrix is given as

$$W = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0086 \end{bmatrix} \tag{6.12}$$

The corresponding discrete system for (6.11) was found for $T = .15$ by using a digital computer. For this, a common NPS subroutine called "PHIDEL" was used which essentially computes the series for e^{AT}. This resulted in:

$$\Phi(.15) = \begin{bmatrix} .994 & .143 & .0065 \\ -.0977 & .884 & .0653 \\ -.979 & -1.207 & .101 \end{bmatrix} \tag{6.13}$$

Only one state of the signal model is observed and

$$H = [1 \quad 0 \quad 0] \tag{6.14}$$

The observations are contaminated by additive white noise whose covariance matrix is

$$R = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix} \tag{6.15}$$

This completes the description of the signal model.

The first step in the design of a suboptimal filter is to find a Lyapunov function for the signal model.

Assuming that $C = I$ the Q matrix of the Lyapunov function for this system was found using the algorithm developed in Appendix B. For this problem the algorithm converged in 42 iterations to a stopping criterion of

$$\max_{ij} \left| d_{ij}(k) \right| < .0005 \tag{6.16}$$

where $d_{ij}(k)$ is an element of the k^{th} correction matrix. See Appendix B for further explanation.

The resulting Q is:

$$Q = \begin{bmatrix} 13.9 & 7.56 & 0.357 \\ 7.56 & 13.6 & 0.789 \\ 0.357 & 0.789 & 1.08 \end{bmatrix} \tag{6.17}$$

In order to design the filter, G must have the $n \times 1$ form

$$G = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \tag{6.18}$$

and equation (6.7) must be solved. Substituting from (6.17) into (6.7) and solving for g_2 and g_3 results in

$$\begin{bmatrix} 13.6 & 0.789 \\ 0.789 & 1.08 \end{bmatrix} \begin{bmatrix} g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} -7.56 \\ -0.357 \end{bmatrix} g_1 \tag{6.19}$$

$$\begin{bmatrix} g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} 0.768 & -0.0561 \\ -0.561 & 0.968 \end{bmatrix} \begin{bmatrix} -7.56 \\ -0.357 \end{bmatrix} g_1$$

$$\begin{aligned} g_2 &= -0.56\, g_1 \\ g_3 &= 0.055\, g_1 \end{aligned} \tag{6.20}$$

The constraining relation (6.9) is used to obtain a restriction on g_1. For this example (6.9) becomes

$$
\begin{aligned}
13.9\, g_1 + 7.56\, g_2 + 0.357\, g_3 \ \geq \ & 13.9\, g_1^{\,2} + 2(7.56)\, g_1 g_2 \\
& + 2(0.357)\, g_1 g_3 \\
& + 13.6\, g_2^{\,2} \\
& + 2(0.789)\, g_2 g_3 \\
& + 1.08\, g_3^{\,2}
\end{aligned}
$$

Substituting the relations (6.20) and reducing this equation yields

$$
\begin{aligned}
9.69\, g_1 &\geq 9.69\, g_1^{\,2} \\
g_1 &\leq 1 \qquad \text{for } g_1 > 0 \qquad (6.21)
\end{aligned}
$$

Thus if g_1 is constrained by equation (6.21) for all n, and g_2, g_3 are determined by (6.20), the filter is sure to be stable by its construction using the sufficient conditions of the Lyapunov method in conjunction with the Hadamard-Gerschgorin Theorem.

The simplest kind of filter to implement is one with constant gains. A simple time varying gain which satisfies (6.21) is:

$$g_1 = \frac{1}{n}, \quad \text{for } n = 1,2,3,\ldots$$

As discussed in Section VI A, a comparison of these filters with the optimal filter can be made by plotting $tr(QP(n))$ for each filter.

The covariance of error matrix $P(n)$ was computed on a digital computer for the optimal filter and for the following two suboptimal gains established by (6.20) and (6.21).

$$G_1 = \begin{bmatrix} 1 \\ -.56 \\ .055 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1/n \\ -.56/n \\ .055/n \end{bmatrix} \qquad (6.22)$$

Computer subroutines for these calculations appear in the Computer Program Section. The subroutine "GAIN" was used for the optimal filter and "SUBCOV" for the filter gains in (6.22). Then using the Lyapunov matrix, Q , in (6.17)

$$E[V] = tr(QP(n)) \tag{6.23}$$

was plotted on the same axes for all three filters as shown in Figure 4.

It is interesting to note that all three filters are stable, and the optimal filter does converge faster than the suboptimal filters as predicted in Chapter V, Section B. The optimal filter has its greatest advantage in the first one or two sample periods. That is, the corrected covariance of error (P) is changing more rapidly here than for the suboptimal filters.

C. CONCLUSIONS

The main developments in this thesis show that it is feasible to design suboptimal filters which do perform nearly as well as the Kalman filter. This is of great advantage because suboptimal filters often lead to a much simpler implementation. Such filters are designed so that they will be ASIL and converge faster than a known bound. The design is accomplished by using Lyapunov functions and requires that the signal model be ASIL.

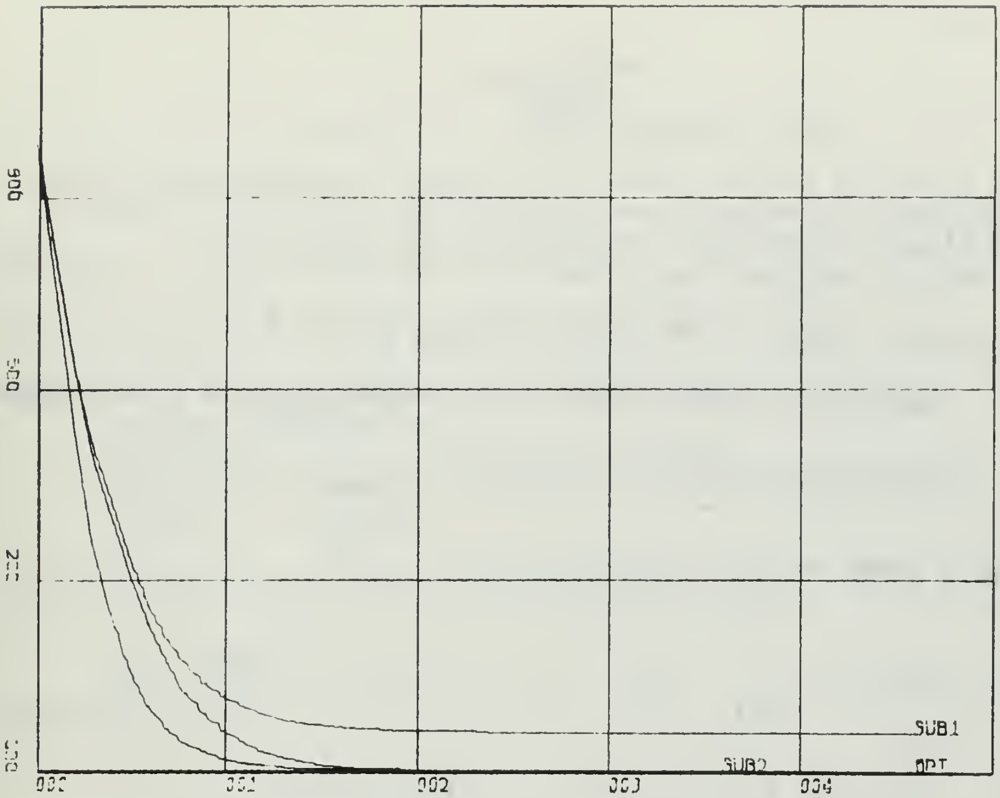
There are two areas brought out by this thesis which require further research.

1. Find a suitable bounding function for the time derivative of the Lyapunov function (4.22 or 4.23) for the optimal filters.

2. Find bounding relations for the covariance of error matrix, P , similar to those developed in Theorem 3.5 for the system state.

The two most important properties of the filter-observer system are its convergence rate and its ability to discriminate against noise. By combining any results obtained for 1. and 2. above, with the techniques developed in Chapter IV of this thesis, a design procedure for sub-optimal filters could be formulated which would insure that the filter be stable and also give information about its two most important properties. Hence, such designs could be formulated and compared with other such designs without the necessity of simulating each filter.

V (20 units per inch)



n (10 units per inch)

FIGURE 4

Lyapunov Function for Various
Filter-Observers

Curve	Filter gain
OPT	OPTIMAL
SUB1	$G_1 = (1 \quad -.56 \quad .055)^T$
SUB2	$G_2 = (1/n \quad -.56/n \quad .055/n)^T$

APPENDIX A

NORMS

A norm is a scalar function of several variables which satisfy the following conditions. The norm is denoted $||\cdot||$.

i) $||X|| \geq 0$ and $||X|| = 0$ iff $X = 0$

ii) $||\omega X|| = |\omega|\ ||X||$ where $|\omega|$ is the magnitude of a complex scalar

iii) $||X + Y|| \leq ||X|| + ||Y||$ Minkowski inequality

Possible norms for an n vector x

$$||x|| = \sum_{i=1}^{n} |x_i| \qquad ||x|| = \sup_{1\leq i\leq n} |x_i|$$

$$||x|| = \left(\Sigma\, |x_i|^2 \right)^{1/2} = (\mathrm{tr}(xx^T))^{1/2}$$

$$||x|| = \left(\Sigma\, |x_i|^P \right)^{1/P} \qquad 1 < P < \infty$$

Possible norms for a general matrix A

$$||A|| = [\mathrm{tr}(AA^T)]^{1/2} \qquad ||A|| = \sum_{i,j} |a_{ij}|$$

$$||A|| = \max_i \sum_{j=1}^{n} |a_{ij}| \qquad ||A|| = \sum_{ij} |a_{ij}|^2 \quad {}^{1/2}$$

$$||A|| = \sup_{||x|| = 1} ||Ax||$$

Other properties — Schwarz inequality

$$||AB|| \leq ||A|| \times ||B|| \qquad ||Ax|| \leq ||A|| \times ||x||$$

Further details may be found in Ref. K1, S1, T1.

APPENDIX B
Lyapunov Function for Linear Discrete System

Davison and Man [D3] have derived an algorithm to numerically generate a Lyapunov function for linear time invariant systems given a negative definite quadratic form for $\dot{V}$. Here a similar algorithm is developed for use with linear discrete systems.

In section III C it is shown that for the discrete system:

$$x(n+1) = A_D x(n) \tag{1}$$

a positive definite Lyapunov function of the form:

$$V = x^T Q x \tag{2}$$

exists with:

$$\Delta V = -x^T C x \tag{3}$$

being negative definite. If the system (1) is stable, Q is determined for any positive definite C from the relation

$$-C = A_D Q A_D - Q \tag{4}$$

In general it is difficult to solve equation (4) for Q even given that $C = I$ let alone some arbitrary positive definite C. So another approach is taken and assumes that the system described by (1) is ASIL.

Proceeding, we write the Lyapunov function following the system trajectory described by (1) as:

$$V(n) = V(0) + V(1) - V(0) + V(2) - V(1) + \ldots + V(n) - V(n-1)$$

since $\Delta V(k) = V(k) - V(k-1)$

then $$V(n) = V(0) + \sum_{i=0}^{n} \Delta V(i) \tag{5}$$

Now let n approach infinity then V(n) approaches zero by assumption that (1) is ASIL. Then (5) becomes:

$$V(0) = -\sum_{i=0}^{\infty} \Delta V(i)$$

Using equation (3)

$$V(0) = \sum_{i=0}^{\infty} x_i^T C x_i \qquad (6)$$

Noting that the solution to (1) is: $x_i = A_D^i x_o$ and that $V(0) = x_o^T Q x_o$

We rewrite equation (6):

$$x_o^T Q x_o = \sum_{i=0}^{\infty} x_o^T (A_D^i)^T C A_D^i x_o$$

which implies that:

$$Q = \sum_{i=0}^{\infty} (A_D^i)^T C A_D^i$$

this summation will converge if equation (1) is stable, since

$|\lambda(A_D)| < 1$ and will not converge if (1) is unstable.

The proposed algorithm as stated here is easily implemented on any computer system having appropriate matrix subroutines.

$$D_k = A_D^T D_{k-1} A_D$$

$$Q_{k+1} = Q_k + D_k \qquad k = 0,1,2,3,...$$

with initial conditions:

$$Q_o = C \qquad D_{-1} = C$$

and termination criterion:

$$||D_{k+1}|| < \varepsilon$$

The matrix, D, is the correction matrix and a simple convergence criterion is to terminate the iteration when the magnitude

of all elements of the correction matrix are less than some desired amount. That is:

$$\max_{ij} |d_{ij}| \leq \varepsilon \quad \text{where} \quad D = (d_{ij})$$

Subroutine "LYAP", given in the computer program section following, was used to generate the Lyapunov Q matrix for C = I in the third order example given in chapter VI. The result is given in equation (6.17).

```
C      THIS SUBROUTINE COMPUTES THE OPTIMUM GAIN MATRIX AND
C      THE ERROR COVARIANCE

       SUBROUTINE GAIN(PKK,PKKM1,Q,R,PHI,H,N,M,G,HI,ND,MD,LD)
      0DIMENSION PKK(12,12),Q(12,12),H(12,12)
      1 ,G(12,12),R(12,12),HI(12,12),HT(12,12),TEMP(12,12),
      2,TEMP1(12,12),PHI(12,12),PHIT(12,12),PKKM1(12,12)
      3TEMP2(12,12)
C      G(K) = P(K/K-1)*HT*(H*P(K/K-1)*HT + R)**(-1)
       CALL TRANS(H,N,N,HT,ND,MD)
       CALL PROD(PKKM1,HT,N,N,N,TEMP,ND,MD,LD)
       CALL PROD(H,TEMP,N,N,N,TEMP1,ND,MD,LD)
       CALL ADD(TEMP1,R,N,N,TEMP1,ND,MD)
       CALL RECIP(M,0.000001,TEMP1,TEMP2,KER,MD)
       IF (KER-2) 101,110,101
   110 WRITE(6,111)
   111 FORMAT (5HKER=2)
   101 CALL PROD(TEMP,TEMP2,N,N,N,G,ND,MD,LD)
C      NOTE HERE  PKK(I,J) = P(K/K)  WHERE
C      P(K/K) = (I-G(K)*H)*P(K/K-1)
       CALL PROD(G,H,N,N,N,TEMP,ND,MD,LD)
       DO 108 I=1,N
       DO 108 J=1,N
   108 TEMP(I,J)=-TEMP(I,J)
       CALL ADD (HI,TEMP,N,N,TEMP,ND,MD)
       CALL PROD(TEMP,PKKM1,N,N,N,PKK,ND,MD,LD)
C      NOTE HERE  PKKM1(I,J) = P(K/K-1)  WHERE
C      P(K/K-1)= PHI*P(K-1/K-1)*PHIT + Q
       CALL TRANS(PHI,N,N,PHIT,ND,MD)
       CALL PROD(PKK,PHIT,N,N,N,TEMP,ND,MD,LD)
       CALL PROD(PHI,TEMP,N,N,N,TEMP1,ND,MD,LD)
       CALL ADD(TEMP1,Q,N,N,PKKM1,ND,MD)
       RETURN
       END


       SUBROUTINE SUBCOV(PKM1,W,R,PHI,H,N,M,G,HI,ND,MD,PK)
       DIMENSION PK(12,12),PKM1(12,12),W(12,12),R(12,12),
      &H(12,12),G(12,12),HI(12,12),TEM(12,12),D(12,12),
      #PHI(12,12),PHIT(12,12)
C     PK=(I-GH)PKM1(I-GH)T+GRGT
       CALL PROD(G,H,N,N,N,TEM,ND,ND,ND)
       CALL SUB(HI,TEM,N,N,TEM,ND,ND)
       CALL TRANS(TEM,N,N,PHIT,ND,ND)
       CALL PROD(TEM,PKM1,N,N,N,D,ND,ND,ND)
       CALL PROD(D,PHIT,N,N,N,PK,ND,ND,ND)
       CALL TRANS(G,N,N,D,ND,ND)
       CALL PROD(G,R,N,N,N,TEM,ND,ND,ND)
       CALL PROD(TEM,D,N,N,N,PHIT,ND,ND,ND)
       CALL ADD(PK,PHIT,N,N,PK   ,N,ND)
C   PKM1=PHI*PK*PHIT+W
       CALL TRANS(PHI,N,N,PHIT,ND,ND)
       CALL PROD(PK   ,PHIT,N,N,N,TEM,ND,ND,ND)
       CALL PROD(PHI,TEM,N,N,N,PKM1,ND,ND,ND)
       CALL ADD(PKM1,W,N,N,PKM1,ND,ND)
       RETURN
       END
```

```
      SUBROUTINE LYAP(SPHI,QL,N,ND,MD,C)
      DIMENSION SPHI(12,12),QL(12,12),C(12,12),PHIT(12,12),
     #D1(12,12),D2(12,12),X2(900)
C   LYAPUNOV FUNCTION
      DO 2004 I=1,N
      DO 2004 J=1,N
      D2(I,J)=C(I,J)
 2004 QL(I,J)=C(I,J)
      CALL TRANS(SPHI,N,N,PHIT,ND,ND)
      DO 2005 K=1,500
      CALL PROD(PHIT,D2,N,N,N,D1,ND,ND,ND)
      CALL PROD(D1,SPHI,N,N,N,D2,ND,ND,ND)
      CALL ADD(QL,D2,N,N,QL,ND,ND)
      TST=0.
      DO 2006 I=1,N
      DO 2006 J=1,N
      TIP=ABS(D2(I,J))
 2006 CONTINUE
      IF(TIP.GT.TST)  TST=TIP
      IF(TST.LE.0.0005) GO TO 2007
 2005 CONTINUE
 2007 WRITE(6,2008) K
 2008 FORMAT(' ITTERATION K=',I3)
      RETURN
      END
```

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13. ABSTRACT

The basic filter-observer equations of Kalman for optimal and suboptimal filters are studied using the concepts of Lyapunov functions and stability theory. The Second Method of Lyapunov is used to form a basis for comparison of the convergence rates of such filters. Lyapunov functions are also used to derive constraining relations for the elements of the filter gain matrix leading to design criteria for suboptimal filters. A derivation of the optimal filter gain based upon the Lyapunov function of a random variable is given. This derivation shows that the optimal filter converges most rapidly. A design of a suboptimal filter for one class of signal models is given based solely upon stability constraints.

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